3-4-1994

Undetermined Coefficients.

A **linear constant coefficient equation** is an equation of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f(x),$$

where a_0, a_1, ... , a_n are numbers.

If $f(x) \neq 0$, the equation is **nonhomogeneous**. The problem is to find the general solution of the equation.

You can find the general solution y_h to the corresponding **homogeneous** equation by using the **characteristic equation**. Here's an example to refresh your memory.

Example. $y'' - y = 0$.

The characteristic equation $m^2 - 1 = 0$ has roots $m = 1$ and $m = -1$. The solution is

$$y = c_1 e^x + c_2 e^{-x}. \quad \square$$

Example. $(D - 3)^4 y = 0$.

The characteristic equation is $(m - 3)^4 = 0$, so $m = 3$ is a quadruple root. The solution is

$$y = c_1 e^{3x} + c_2 x e^{3x} + c_3 x^2 e^{3x} + c_4 x^3 e^{3x}. \quad \square$$

Example. $y'' - 4y' + 5y = 0$.

The characteristic equation is $m^2 - 4m + 5 = 0$. The roots are $m = 2 \pm i$. The solution is

$$y = c_1 e^{2x} \cos x + c_2 e^{2x} \sin x. \quad \square$$

Return to the original problem. Suppose that y_p is a **particular** solution to the original equation. That is,

$$a_n y_p^{(n)} + a_{n-1} y_p^{(n-1)} + \ldots + a_1 y_p' + a_0 y_p = f(x).$$

Then the general solution to the original equation is $y = y_h + y_p$.

To understand why this **superposition** works, write the equation using operator notation:

$$(a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0) y = f(x).$$

To simplify the notation, write

$$p(D) \quad \text{for} \quad a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0.$$

Therefore,

$$p(D) y_h = 0, \quad p(D) y_p = f(x).$$

Add the equations, using the fact that $p(D)$ is a **linear** operator:

$$p(D)(y_h + y_p) = f(x).$$

Therefore, $y_h + y_p$ *is* a solution to the equation.
Moreover, it is the *general* solution: If y is any solution to the equation, then

$$p(D)y = f(x) \quad \text{and} \quad p(D)y_p = f(x).$$

Subtracting the second equation from the first,

$$p(D)(y - y_p) = 0.$$

Since $y - y_p$ is a homogeneous solution, and since y_h is the *general* homogeneous solution, I can obtain $y - y_p$ by appropriate choice of the constants in y_h. Assuming that I've done so, I get $y - y_p = y_h$, or $y = y_h + y_p$.

Example. Solve $y'' - y = -6\cos x$.

The general solution to the homogeneous equation was $y_h = c_1 e^x + c_2 e^{-x}$. $y_p = 3\cos x$ is a particular solution. Check: The derivatives are

$$y_p = 3\cos x, \quad y_p' = -3\sin x, \quad y_p'' = -3\cos x.$$

Therefore,

$$y_p'' - y_p = -3\cos x - 3\cos x = -6\cos x.$$

The equation is satisfied.
By the discussion above, the general solution to $y'' - y = -6\cos x$ is

$$y = c_1 e^x + c_2 e^{-x} + 3\cos x. \ \square$$

Thus, solving a linear constant coefficient equation reduces to the problem of finding a particular solution. The **method of undetermined coefficients** produces a particular solution y_p to

$$(a_n D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0)y = f(x)$$

if $f(x)$ is a solution to a linear homogeneous constant coefficient equation $q(D)f(x) = 0$.
This stipulation means that $f(x)$ can be a sum of products of terms involving

$$e^{ax} \qquad \text{or} \qquad p_n x^n + \ldots + p_1 x + p_0 \qquad \text{or} \qquad \left\{ \begin{matrix} \cos bx \\ \sin bx \end{matrix} \right\}.$$

For example, you *can* apply undetermined coefficients to any of these equations:

$$y'' + y = e^{2x} + 1, \qquad y''' - y'' = 30 - 30x, \qquad (D^2 - 4D + 4)y = e^x \cos 2x.$$

You *cannot* apply undetermined coefficients to any of these equations:

$$y'' + y = \tan x, \qquad (D-1)^2 y = \frac{e^x}{x}, \qquad y'' - 2y' - 3y = \sqrt{x}.$$

(For the last three equations, you will need another method, such as **variation of parameters**.)

Assume that $f(x)$ has the appropriate form, so there is an operator polynomial $q(D)$ such that

$$q(D)f(x) = 0.$$

I want a particular solution y_p to the original equation, which means that

$$p(D)y_p = f(x).$$

Apply $q(D)$ to both sides of this equation:

$$p(D)q(D)y_p = q(D)\,[f(x)] = 0.$$

This is a *homogeneous* linear constant coefficient equation, so I can solve it for y_p.

However, this isn't the whole story. At this point, y_p will contain some arbitrary constants. I need to find values for these constants — the **undetermined coefficients** — which will make y_p a particular solution. To do this I'll plug the expression for y_p back into the original equation $p(D)y = f(x)$.

Before doing this plug in, I'll save myself some work: If y_p contains "pieces" which are homogeneous solutions, I'll throw them out before plugging y_p into $p(D)y = f(x)$. (Being homogeneous solutions, these pieces will simply give 0 when they're operated on by $p(D)$.)

Whew! This sounds kind of involved, but it's actually pretty straightforward (if a bit tedious), as you'll see in the examples that follow.

Example. Solve $y'' + y = e^{2x} + 1$.

Write the equation in operator form:

$$(D^2 + 1)y = e^{2x} + 1.$$

The homogeneous solution is

$$y_h = c_1 \cos x + c_2 \sin x.$$

A particular solution y_p must satisfy

$$(D^2 + 1)y_p = e^{2x} + 1.$$

I want to apply an operator polynomial which will kill the right side. e^{2x} is a solution to $(D-2)y = 0$; 1 is a solution to $Dy = 0$. Hence, I apply $D(D-2)$ to both sides, and obtain

$$D(D-2)(D^2+1)y_p = D(D-2)(e^{2x}+1) = 0.$$

It follows that

$$y_p = A + Be^{2x} + C\cos x + D\sin x.$$

(I've used A, B, C, and D to avoid confusion with the arbitrary constants in the homogeneous solution.) Throw out $C\cos x$ and $D\sin x$, since they're homogeneous solutions. This leaves

$$y_p = A + Be^{2x}.$$

Next, plug y_p into the original equation. Compute the derivatives:

$$y_p' = 2Be^{2x}, \quad y_p'' = 4Be^{2x}.$$

Then plugging in yields

$$y_p'' + y_p = 4Be^{2x} + (A + Be^{2x}) = A + 5Be^{2x} = e^{2x} + 1.$$

Equate corresponding coefficients. I get $A = 1$, and $5B = 1$ so $B = \frac{1}{5}$.
Therefore, $y_p = 1 + \frac{1}{5}e^{2x}$.
The general solution is

$$y = y_h + y_p = c_1 \cos x + c_2 \sin x + 1 + \frac{1}{5}e^{2x}. \quad \square$$

Example. Solve $y''' - y'' = 30 - 30x$.

Write the equation in operator form:

$$D^2(D-1)y = 30 - 30x.$$

The homogeneous solution is

$$y_h = c_1 + c_2 x + c_3 e^x.$$

A particular solution y_p must satisfy

$$D^2(D-1)y_p = 30 - 30x.$$

I want to apply an operator polynomial which will kill the right side. $30 - 30x$ is killed by D^2 (as you can check by differentiating it twice). Hence, I apply D^2 to both sides, and obtain

$$D^4(D-1)y_p = 0.$$

Therefore,

$$y_p = A + Bx + Cx^2 + Dx^3 + Ee^x.$$

Throw out $A + Bx$ and Ee^x, since they're homogeneous solutions. This leaves

$$y_p = Cx^2 + Dx^3.$$

Next, plug y_p into the original equation. Compute the derivatives:

$$y_p' = 2Cx + 3Dx^2, \quad y_p'' = 2C + 6Dx, \quad y_p''' = 6D.$$

Plug in:

$$y_p''' - y_p'' = 6D - (2C + 6Dx) = (6D - 2C) - 6Dx = 30 - 30x.$$

Equating corresponding coefficients, I obtain $6D - 2C = 30$ and $-6D = -30$. The second equation gives $D = 5$; plugging this into the first equation produces $C = 0$.
Therefore, $y_p = 5x^3$.
The general solution is

$$y = y_h + y_p = c_1 + c_2 x + c_3 e^x + 5x^3. \quad \square$$

Example. Solve $(D^2 - 4D + 4)y = e^x \cos 2x$.

The equation may be written as

$$(D-2)^2 y = e^x \cos 2x.$$

The homogeneous solution is

$$y_h = c_1 e^{2x} + c_2 x e^{2x}.$$

In situations like this one where the right side involves trigonometric functions, it's often possible to simplify the computations by using **complex exponentials**.

Recall the fundamental identity

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

There is a $e^x \cos 2x$ on the right side, and

$$e^x \cos 2x = \operatorname{re} e^x (\cos 2x + i \sin 2x) = \operatorname{re} e^x e^{2xi} = \operatorname{re} e^{(1+2i)x}.$$

Therefore, replace the original equation with

$$(D-2)^2 y = e^{(1+2i)x}.$$

I'll find a particular solution for *this* equation, then take the real part to obtain a particular solution for the *original* equation — the *real* part because $e^x \cos 2x = \operatorname{re} e^{(1+2i)x}$.

A particular solution y_p must satisfy

$$(D-2)^2 y_p = e^{(1+2i)x}.$$

I want to apply an operator polynomial which will kill the right side. $e^{(1+2i)x}$ is killed by $D-(1+2i)$. Hence, I apply $D-(1+2i)$ to both sides, and obtain

$$(D-(1+2i))\,(D-2)^2 y_p = 0.$$

Therefore,

$$y_p = Ae^{(1+2i)x} + Be^{2x} + Cxe^{2x}.$$

Throw out Be^{2x} and Cxe^{2x}, since they're homogeneous solutions. Thus,

$$y_p = Ae^{(1+2i)x}.$$

(Note that A is a *complex* coefficient!)

Next, plug y_p into the original equation. Compute the derivatives:

$$y_p' = A(1+2i)e^{(1+2i)x}, \quad y_p'' = A(1+2i)^2 e^{(1+2i)x} = A(-3+4i)e^{(1+2i)x}.$$

Plug in:

$$y_p'' - 4y_p' + 4y_p = A(-3+4i)e^{(1+2i)x} - 4A(1+2i)e^{(1+2i)x} + 4Ae^{(1+2i)x} = -A(3+4i)e^{(1+2i)x} = e^{(1+2i)x}.$$

Now $-A(3+4i) = 1$, so

$$A = -\frac{1}{3+4i} = -\frac{1}{3+4i}\,\frac{3-4i}{3-4i} = -\frac{3}{25} + \frac{4}{25}i.$$

(It's important to simplify A to $a+bi$ form!)

Therefore,

$$y_p = \left(-\frac{3}{25} + \frac{4}{25}i\right) e^{(1+2i)x} = \left(-\frac{3}{25} + \frac{4}{25}i\right) e^x e^{2xi} =$$

$$\frac{1}{25}e^x(-3+4i)(\cos 2x + i\sin 2x) = \frac{1}{25}e^x\left((-3\cos 2x - 4\sin 2x) + i(4\cos 2x - 3\sin 2x)\right).$$

I had to simplify enough so that I could recognize the real and imaginary parts of y_p.

The real part gives a particular solution to the original equation:

$$\text{“actual” } y_p = \frac{1}{25}e^x(-3\cos 2x - 4\sin 2x).$$

(To repeat, I take the *real* part because $\cos 2x = \operatorname{re}(e^{2xi})$. If the original equation had a $\sin 2x$ on the right instead, I'd have to take the *imaginary* part.)

Therefore, the general solution is

$$y = y_h + y_p = c_1 e^{2x} + c_2 x e^{2x} + \frac{1}{25} e^x(-3\cos 2x - 4\sin 2x). \ \square$$

Problems on Undetermined Coefficients.

1. Find the general solution to $y'' - y = xe^x$.

2. Solve the initial value problem $(D-3)^2 y = 12xe^{3x}$, $y(0) = 2$, $y'(0) = 4$.

3. Find the general solution to $y'' - y' - 2y = e^{2x}\sin 2x$.

4. Calvin Butterball is having trouble with the differential equation

$$y'' + y = \sec x.$$

"I tried $y_p = A\sec x$, but it doesn't work," he grumbles. "When I solve for A, I get $A = \frac{1}{2}(\cos x)^2$, so $y_p = \frac{1}{2}\cos x$. But it doesn't check when I plug it back into the equation!"

Phoebe Small, Calvin's paramour and math tutor, rolls her eyes. "Calvin, you can't use undetermined coefficients on that equation! The right side is $\sec x$, and that's not a solution to a linear constant coefficient homogeneous equation!"

Calvin frowns. "Then how am I supposed to solve the equation?"

Well? Are you going to tell him or aren't you?

Solutions to the Problems on Undetermined Coefficients.

1. Find the general solution to $y'' - y = xe^x$.

Write the equation in operator form:

$$(D-1)(D+1)y = xe^x.$$

The homogeneous solution is

$$y_h = c_1 e^x + c_2 e^{-x}.$$

A particular solution y_p must satisfy

$$(D-1)(D+1)y_p = xe^x.$$

I want to apply an operator polynomial which will kill the right side. xe^x is a solution to $(D-1)^2 y = 0$. Hence, I apply $(D-1)^2$ to both sides, and obtain

$$(D-1)^3(D+1)y_p = 0.$$

It follows that

$$y_p = Ae^x + Bxe^x + Cx^2e^x + De^{-x}.$$

Ae^x and De^{-x} are homogeneous solutions: throw them out. This leaves

$$y_p = Bxe^x + Cx^2e^x.$$

Next, plug y_p into the original equation. Compute the derivatives:

$$y_p' = Be^x + Bxe^x + 2Cxe^x + Cx^2e^x, \quad y_p'' = 2Be^x + 2Ce^x + Bxe^x + 4Cxe^x + Cx^2e^x.$$

Plug in:

$$y_p'' - y_p = 4Cxe^x + (2B + 2C)e^x = xe^x.$$

Equate corresponding coefficients. I get $4C = 1$ and $2B + 2C = 0$. The first equation gives $C = \frac{1}{4}$; plugging this into the second, I obtain that $B = -\frac{1}{4}$.

Therefore, $y_p = \frac{1}{4}x^2e^x - \frac{1}{4}xe^x$.

The general solution is

$$y = y_h + y_p = c_1e^x + c_2e^{-x} + \frac{1}{4}x^2e^x - \frac{1}{4}xe^x. \square$$

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2. Solve the initial value problem $(D-3)^2y = 12xe^{3x}$, $y(0) = 2$, $y'(0) = 4$.

The homogeneous solution is

$$y_h = c_1e^{3x} + c_2xe^{3x}.$$

A particular solution y_p must satisfy

$$(D-3)^2y_p = 12xe^{3x}.$$

I want to apply an operator polynomial which will kill the right side. $12xe^{3x}$ is a solution to $(D-3)^2y = 0$. Hence, I apply $(D-3)^2$ to both sides, and obtain

$$(D-3)^4y_p = 0.$$

Therefore,

$$y_p = Ae^{3x} + Bxe^{3x} + Cx^2e^{3x} + Dx^3e^{3x}.$$

Ae^{3x} and Bxe^{3x} are homogeneous solutions. Throw them out, leaving

$$y_p = Cx^2e^{3x} + Dx^3e^{3x}.$$

Next, plug y_p into the original equation. Compute the derivatives:

$$y_p' = 2Cxe^{3x} + 3Cx^2e^{3x} + 3Dx^2e^{3x} + 3Dx^3e^{3x},$$

$$y_p'' = 2Ce^{3x} + 12Cxe^{3x} + 6Dxe^{3x} + 9Cx^2e^{3x} + 18Dx^2e^{3x} + 9Dx^3e^{3x}.$$

Plug in:

$$y_p'' - 6y_p' + 9y_p = 2Ce^{3x} + 6Dxe^{3x} = 12xe^{3x}.$$

Equate corresponding coefficients: $2C = 0$ and $6D = 12$, so $C = 0$ and $D = 2$.

Therefore, $y_p = 2x^3e^{3x}$.

The general solution is

$$y = y_h + y_p = c_1e^{3x} + c_2xe^{3x} + 2x^3e^{3x}.$$

Oop! I almost forgot to plug in the initial conditions! $2 = y(0) = c_1$, so

$$y = 2e^{3x} + c_2 x e^{3x} + 2x^3 e^{3x}.$$

Differentiate:

$$y' = 6e^{3x} + c_2 e^{3x} + 3c_2 x e^{3x} + 6x^2 e^{3x} + 6x^3 e^{3x}.$$

Now $4 = y'(0) = 6 + c_2$, so $c_2 = -2$.
The solution is

$$y = 2e^{3x} - 2xe^{3x} + 2x^3 e^{3x}. \ \square$$

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3. Find the general solution to $y'' - y' - 2y = e^{2x} \sin 2x$.

In operator form, the equation is

$$(D-2)(D+1)y = e^{2x} \sin 2x.$$

The homogeneous solution is

$$y_h = c_1 e^{2x} + c_2 x e^{-x}.$$

Now

$$e^{2x} \sin 2x = \operatorname{im} e^{2x} (\cos 2x + i \sin 2x) = \operatorname{im} e^{2x} \cdot e^{2xi} = \operatorname{im} e^{(2+2i)x}.$$

So complexify the equation to

$$(D-2)(D+1)y = e^{(2+2i)x}.$$

I will find a particular solution for *this* equation, then take the imaginary part to obtain a particular solution for the *original* equation.

A particular solution y_p must satisfy

$$(D-2)(D+1)y_p = e^{(2+2i)x}.$$

I want to apply an operator polynomial which will kill the right side. $e^{(2+2i)x}$ is killed by $D - (2+2i)$. Hence, I apply $D - (2+2i)$ to both sides, and obtain

$$(D - (2+2i))(D-2)(D+1)y_p = 0.$$

It follows that

$$y_p = Ae^{(2+2i)x} + Be^{2x} + Ce^{-x}.$$

Throw out Be^{2x} and Ce^{-x}, since they're homogeneous solutions. This leaves

$$y_p = Ae^{(2+2i)x}.$$

Next, plug y_p into the original equation. Compute the derivatives:

$$y_p' = A(2+2i)e^{(2+2i)x}, \quad y_p'' = A(2+2i)^2 e^{(2+2i)x} = 8Aie^{(2+2i)x}.$$

Plug in:

$$y_p'' - y_p' - 2y_p = 8Aie^{(2+2i)x} - A(2+2i)e^{(2+2i)x} - 2Ae^{(2+2i)x} = A(-4+6i)e^{(2+2i)x} = e^{(2+2i)x}.$$

Now $A(-4+6i) = 1$, so

$$A = \frac{1}{-4+6i} = \frac{1}{-4+6i} \frac{-4-6i}{-4-6i} = -\frac{1}{26}(2+3i).$$

Therefore,

$$y_p = -\frac{1}{26}(2+3i)e^{(1+2i)x} = -\frac{1}{26}(2+3i)e^{2x}e^{2xi} =$$

$$\frac{1}{26}(2-3i)e^{2x}(\cos 2x + i\sin 2x) = -\frac{1}{26}e^{2x}\left((2\cos 2x - 3\sin 2x) + i(3\cos 2x + 2\sin 2x)\right).$$

The imaginary part gives a particular solution to the original equation:

$$\text{“actual” } y_p = -\frac{1}{26}e^{2x}(3\cos 2x + 2\sin 2x).$$

Therefore, the general solution is

$$y = y_h + y_p = c_1e^{2x} + c_2e^{-x} - \frac{1}{26}e^{2x}(3\cos 2x + 2\sin 2x). \quad \square$$

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Calvin frowns. "Then how am I supposed to solve the equation?"

Well? Are you going to tell him or aren't you?

Calvin ought to be using variation of parameters instead.

$\cos x$ and $\sin x$ are independent homogeneous solutions, so the general solution is $y = u_1\cos x + u_2\sin x$, where u_1 and u_2 satisfy

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \sec x \end{bmatrix}.$$

(Refer to your notes on variation of parameters if you've forgotten where this comes from.)

Invert the square matrix and put the inverse on the right:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \begin{bmatrix} 0 \\ \sec x \end{bmatrix} = \begin{bmatrix} -\tan x \\ 1 \end{bmatrix}.$$

$u_1' = -\tan x$, so $u_1 = \ln|\cos x| + c_1$. $u_2' = 1$, so $u_2 = x + c_2$. The solution is

$$y = u_1\cos x + u_2\sin x = (\cos x)\ln|\cos x| + c_1\cos x + x\sin x + c_2\sin x.$$

Okay, Calvin? $\square$

