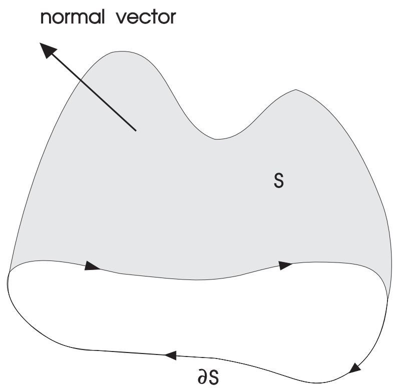
Math 227
10-14-1993

Stokes' theorem.

Stokes' theorem relates the flux of the curl of a vector field through a surface to the circulation of the field around the boundary of the surface:

$$\iint_S \operatorname{curl} \vec{F} \cdot \overrightarrow{dS} = \int_{\partial S} \vec{F} \cdot \overrightarrow{ds}.$$



S is a surface and ∂S is its **boundary**. (If S is a bowl, then ∂S is the rim of the bowl.) I need to assume that S is **orientable** — that is, that I can choose a normal vector to the surface a each point in a "smooth" way.

The boundary must be oriented in a way that is consistent with the choice of the normal to the surface. Imagine walking along the surface near the edge so that your body is parallel to the normal. Your left arm should be toward the "inside" of the surface and your right arm should be toward the "outside".

- The general form of Stokes' theorem for **differential forms** looks like this:

$$\int_{\partial\Sigma} \omega = \int_{\Sigma} d\omega.$$

Special cases: Stokes' theorem in 3 dimensions, the Divergence Theorem, and the Fundamental Theorem of Calculus.

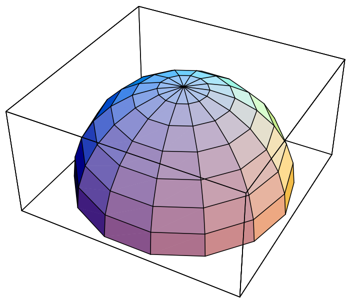
- If $\vec{F} = \langle P(x,y), Q(x,y), 0\rangle$ and S is a region in the plane,

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle.$$

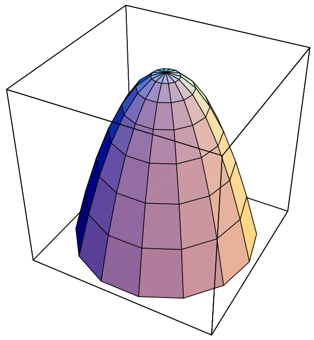
Dotting this with the normal $\langle 0,0,1\rangle$ yields the integrand $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$. In other words, Green's theorem is Stokes' theorem in the x-y plane.

Example. Suppose $\vec{F}$ is a vector field, and S is a surface. Then the flux of $\operatorname{curl} \vec{F}$ through S is equal to the flux of $\operatorname{curl} \vec{F}$ *through any surface that has the same boundary as S.*

For example, consider the hemisphere $x^2 + y^2 + z^2 = 1$ for $z \geq 0$.

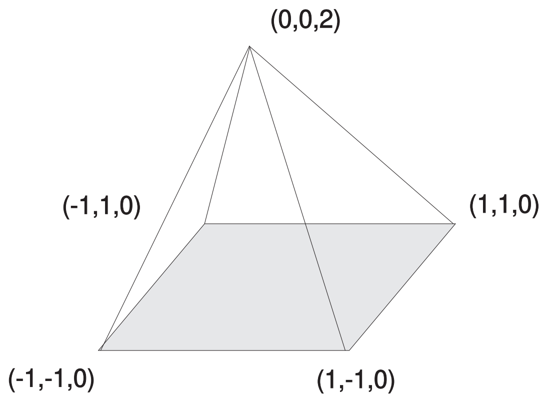


The boundary of the hemisphere is the unit circle $x^2 + y^2 = 1$ for $z = 0$. The unit circle is also the boundary of the paraboloid $z = 2 - 2x^2 - 2y^2$ for $z \geq 0$.



Therefore, if $\vec{F}$ is a vector field, the flux of curl $\vec{F}$ through the hemisphere is the same as the flux through the paraboloid. $\square$

Example. Consider the pyramid with vertices $(-1, -1, 0)$, $(-1, 1, 0)$, $(1, -1, 0)$, $(1, 1, 0)$, and $(0, 0, 2)$.



(a) Let S be the surface consisting of the four triangular faces, and let $\vec{F} = \langle y^2 + z, z^2 - x, x^2 + y \rangle$. Compute the flux of curl $\vec{F}$ upward through S.

To compute the flux directly, you'd parametrize each of the triangular faces, compute the flux of curl $\vec{F}$ through each face, and add up the results: 4 parametrizations and 4 integrals. It is much easier to use Stokes' theorem.

Stokes' theorem says:

$$\iint_S \operatorname{curl} \vec{F} \cdot \overrightarrow{dS} = \int_{\partial S} \vec{F} \cdot \overrightarrow{ds}.$$

The boundary ∂S is the square with vertices $(-1,-1,0)$, $(-1,1,0)$, $(1,-1,0)$, and $(1,1,0)$ traversed in the counterclockwise direction.

To compute the line integral around the boundary, you'd need to parametrize each segment of the boundary, compute the integral of $\vec{F}$ along each piece, and add up the results: 4 parametrizations and 4 integrals again!

Here's a better way. By the remarks in the last example, to compute the flux of curl $\vec{F}$, you can use *any* surface with the same boundary as S. The boundary curve is the square, and there is an "obvious" surface whose edge is the square: The square base B of the tetrahedron. It is given by $z = 0$, for

$$-1 \le x \le 1, \quad -1 \le y \le 1.$$

This surface is a "single piece", as opposed to 4 pieces — in fact, it's the graph of a very simple function ($z = 0$). This means that only one integral is needed to compute the flux.

Since the surface is (part of) the x-y-plane, the normal vector is $\vec{N} = \langle 0,0,1\rangle$. Next,

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 + z & z^2 - x & x^2 + y \end{vmatrix} = \langle 1 - 2z, 1 - 2x, -1 - 2y\rangle.$$

Hence,

$$\operatorname{curl} \vec{F} \cdot \vec{N} = -1 - 2y.$$

The flux is

$$\iint_B \operatorname{curl} \vec{F} \cdot \overrightarrow{dS} = \int_{-1}^{1}\int_{-1}^{1} (-1 - 2y)\, dy\, dx = -4. \square$$

(b) Let S be the surface consisting of the four triangular faces *and* the base of the pyramid. Let $\vec{F} = \langle y^2 + z, z^2 - x, x^2 + y\rangle$. Compute the flux of curl $\vec{F}$ upward through S.

By Stokes' theorem,

$$\iint_S \operatorname{curl} \vec{F} \cdot \overrightarrow{dS} = \int_{\partial S} \vec{F} \cdot \overrightarrow{ds}.$$

However, S is a *closed surface*: It has no boundary. Since $\partial S = \emptyset$,

$$\int_{\partial S} \vec{F} \cdot \overrightarrow{ds} = 0.$$

Therefore,

$$\iint_S \operatorname{curl} \vec{F} \cdot \overrightarrow{dS} = 0.$$

In general, the flux of curl $\vec{F}$ through a closed surface is 0.

Note, however, that the flux of an arbitrary vector field $\vec{F}$ through a closed surface need not be 0. Think of the closed surface as a membrane and $\vec{F}$ as the velocity field of a fluid flow. The flux of $\vec{F}$ out of the surface will be negative (and nonzero) when the fluid is being "compressed" inside the surface. $\square$

Problems on Stokes' theorem.

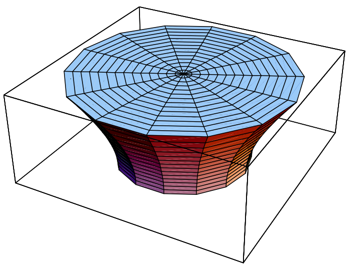
1. Let S be the surface consisting of the disk $x^2+y^2 \leq 25$ in the plane $z=4$ together with the hyperboloid $x^2+y^2-z^2=9$ for $0 \leq z \leq 4$. Let $\vec{F} = \langle -y, x, e^{xyz} \rangle$. Compute the flux of curl $\vec{F}$ outward through S.

2. Compute the circulation of $\vec{F} = \langle xyz, xy, z \rangle$ counterclockwise around the rectangle with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,1)$, and $(0,1,1)$.

3. Compute the circulation of $\vec{F} = \langle x+y, 2x-z, y+z \rangle$ around the triangle with vertices $(2,0,0)$, $(0,3,0)$, $(0,0,6)$, where the path is traversed counterclockwise as viewed from above.

Solutions to the Problems on Stokes' theorem.

1. Let S be the surface consisting of the disk $x^2+y^2 \leq 25$ in the plane $z=4$ together with the hyperboloid $x^2+y^2-z^2=9$ for $0 \leq z \leq 4$. Let $\vec{F} = \langle -y, x, e^{xyz} \rangle$. Compute the flux of curl $\vec{F}$ outward through S.



By Stokes' theorem,

$$\iint_S \operatorname{curl} \vec{F} \cdot \overrightarrow{dS} = \int_{\partial S} \vec{F} \cdot \overrightarrow{ds}.$$

That is, the flux is equal to the circulation of $\vec{F}$ around ∂S.

The boundary of S is the circle $x^2+y^2=9$ in the x-y-plane (as you can see by setting $z=0$ in the equation of the hyperboloid). This curve may be parametrized by

$$\vec{\sigma}(t) = \langle 3\cos t, 3\sin t, 0 \rangle, \quad 0 \leq t \leq 2\pi.$$

Now compute the circulation of $\vec{F}$ around $\vec{\sigma}$:

$$\vec{F}(t) = \langle -3\sin t, 3\cos t, 1 \rangle \quad \text{and} \quad \vec{\sigma}\,'(t) = \langle -3\sin t, 3\cos t, 0 \rangle.$$

Therefore,

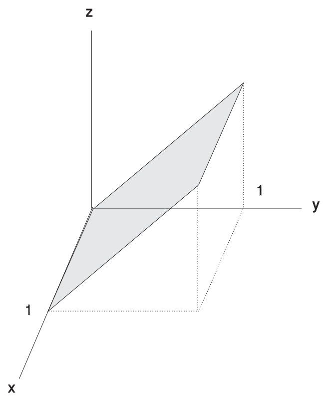
$$\vec{F} \cdot \vec{\sigma}\,'(t) = 9.$$

Hence,

$$\int_{\partial S} \vec{F} \cdot \overrightarrow{ds} = \int_0^{2\pi} 9\, dt = 18\pi \approx 56.54867. \ \square$$

2. Compute the circulation of $\vec{F} = \langle xyz, xy, z \rangle$ counterclockwise around the rectangle with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,1)$, and $(0,1,1)$.

Here is the rectangle:



A direction computation would require 4 line integrals, one for each boundary segment. It is better to use Stokes' theorem:

$$\int_{\partial S} \vec{F} \cdot \overrightarrow{ds} = \iint\limits_{S} \operatorname{curl} \vec{F} \cdot \overrightarrow{dS}.$$

The rectangle lies in the plane $z = y$, and its projection into the x-y-plane is given by

$$0 \le x \le 1, \quad 0 \le y \le 1.$$

Now

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz & xy & z \end{vmatrix} = \langle 0, xy, y - xz \rangle.$$

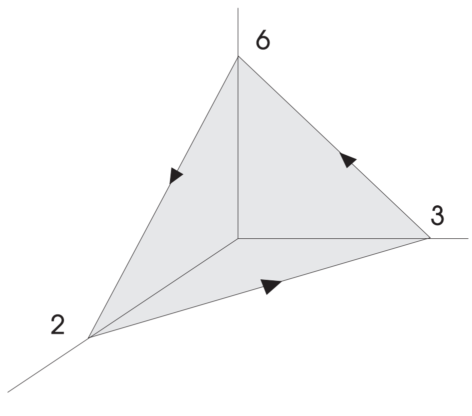
The normal is $\vec{N} = \langle 0, -1, 1 \rangle$, so

$$\operatorname{curl} \vec{F} \cdot \vec{N} = -2xy + y.$$

Therefore,

$$\iint\limits_{S} \operatorname{curl} \vec{F} \cdot \overrightarrow{dS} = \int_0^1 \int_0^1 (-2xy + y)\, dx\, dy = 0. \square$$

3. Compute the circulation of $\vec{F} = \langle x + y, 2x - z, y + z \rangle$ around the triangle with vertices $(2, 0, 0)$, $(0, 3, 0)$, $(0, 0, 6)$, where the path is traversed counterclockwise as viewed from above.



The surface is the part of the plane $3x + 2y + z = 6$ lying in the first octant. To compute the circulation directly, you would need to parametrize each segment and compute 4 line integrals (one over each segment).

By Stokes' theorem, the circulation is equal to the flux of curl $\vec{F}$ upward through the triangle. First,

$$\operatorname{curl} \vec{F} = \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \\ x+y & 2x-z & y+z \end{array} \right| = \langle 2, 0, 1 \rangle.$$

Write the plane as $z = 6 - 3x - 2y$. The *upward* normal is $\vec{N} = \langle 3, 2, 1 \rangle$. Therefore,

$$\operatorname{curl} \vec{F} \cdot \vec{N} = 7.$$

The flux of curl $\vec{F}$ upward through the triangle is

$$\int\!\!\int_T 7\, dx\, dy,$$

where T is the projection of the triangle into the x-y-plane. This double integral is just 7 times the area of T.

Now T is

$$0 \le x \le 2, \quad 0 \le y \le 3 - \frac{3}{2} x.$$

Therefore, the area of T is 3. The flux of curl $\vec{F}$ — and hence the circulation — is $7 \cdot 3 = 21$. □