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Reduction of Order.

Reduction of order produces the general solution to a second-order linear equation

$$y'' + p(x)y' + q(x)y = r(x)$$

given a solution y_1 to the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

The idea is to "guess" that the general solution has the form

$$y = y_1 u,$$

where y_1 is the given homogeneous solution.

(It may seem silly for a solution method to require that you already know a solution. It's not so silly — you may be able to see that a *particular* solution works without knowing the general solution, for instance.)

I'm going to plug $y = y_1 u$ into the original equation. First, I'll compute the derivatives:

$$y' = y_1 u' + y_1' u, \quad y'' = y_1 u'' + 2y_1' u' + y_1'' u.$$

Plug in:

$$\begin{array}{rcccccccc} y'' + p(x)y' + q(x)y & = & y_1 u'' & + & 2y_1' u' & + & y_1'' u & & \\ & & & + & p y_1 u' & + & p y_1' u & & \\ & & & & & + & q y_1 u & & \\ & = & y_1 u'' & + & (2y_1' + p y_1)u' & + & (y_1'' + p y_1' + q y_1)u & = & r(x). \end{array}$$

However,

$$y_1'' + p y_1' + q y_1 = 0,$$

because y_1 was a solution to the homogeneous equation. So

$$y_1 u'' + (2y_1' + p y_1)u' = r(x).$$

Observe that in this equation the "unknown" is u, and that only the derivatives u' and u'' are present. Let $v = u'$, so $v' = u''$. Then

$$y_1 v' + (2y_1' + p y_1)v = r(x).$$

This is the **reduction of order** step; the equation has been reduced from a second-order equation to a first-order equation. This first-order equation may be solved by various methods — for example, by using an integrating factor.

More generally, if you know n independent solutions to the homogeneous equation, you may use a generalization of this method to reduce the order of a linear equation by n.

Example. $y'' - 4y' + 4y = 0$.

In operator notation, this is

$$(D^2 - 4D + 4)y = 0 \quad \text{or} \quad (D-2)^2 y = 0.$$

The characteristic equation $(m-2)^2 = 0$ has a double root $m = 2$. If you've worked with constant coefficient equations, you know that e^{2x} and xe^{2x} are independent solutions, and that the general solution is $y = c_1 e^{2x} + c_2 x e^{2x}$. Reduction of order "explains" why the repeated root produces the "x" in "xe^{2x}".

First, observe that e^{2x} is clearly a solution to $(D-2)^2 y = 0$, since it's a solution to $(D-2)y = 0$. Apply reduction of order, with $y_1 = e^{2x}$. Compute the derivatives:

$$y' = u'e^{2x} + 2ue^{2x}, \quad y'' = u''e^{2x} + 4u'e^{2x} + 4ue^{2x}.$$

Plug these into the original equation:

$$\begin{array}{rclrlrl} y'' - 4y' + 4y & = & u''e^{2x} & + & 4u'e^{2x} & + & 4ue^{2x} \\ & & & - & 4u'e^{2x} & - & 8ue^{2x} \\ & & & & & + & 4ue^{2x} \\ & = & u''e^{2x} & & & & = 0. \end{array}$$

Thus, $u'' = 0$, and you can solve this equation by simply integrating twice:

$$u' = c_1, \quad u = c_1 x + c_2.$$

Now $y = ue^{2x}$, so

$$u = c_1 x e^{2x} + c_2 e^{2x}.$$

This is the solution you'd expect. □

Remarks.

(a) I think it's best to work out the reduction of order computation "by hand" in each problem (as I did in this example). If you attempt to memorize the formula we obtained earlier, you are likely to remember it incorrectly.

(b) You are less likely to make a computational error if you arrange the computation in columns, as I did above. This makes it much easier to add up the terms containing u'', u', and u. □

Example. The **Legendre equation**

$$(1 - x^2)y'' - 2xy' + 2y = 0$$

has $y_1 = x$ as a solution. I'll use reduction of order to find the general solution.

Compute the derivatives:

$$y' = u'x + u, y'' = u''x + 2u'.$$

Plug these expressions into the original equation and solve for u:

$$\begin{array}{rclrlrl} (1 - x^2)y'' - 2xy' + 2y & = & x(1 - x^2)u'' & + & 2(1 - x^2)u' & & \\ & & & - & 2x^2 u' & - & 2xu \\ & & & & & + & 2ux \\ & = & x(1 - x^2)u'' & + & (2 - 4x^2)u' & & = 0. \end{array}$$

Let $v = u'$, so $v' = u''$. Then

$$x(1 - x^2)v' + (2 - 4x^2)v = 0, \quad x(1 - x^2)\frac{dv}{dx} = -2(1 - 2x^2)v.$$

Separate variables:

$$\int \frac{dv}{v} = -2 \int \frac{1 - 2x^2}{x(1 - x^2)}\, dx, \quad \ln|v| = -2\left(\ln|x| + \frac{1}{2}\ln|1 - x^2|\right) + C.$$

Here's how I did the integral on the right:

$$-2\int \frac{1-2x^2}{x(1-x^2)}\,dx = -2\int \frac{1-x^2-x^2}{x(1-x^2)}\,dx = -2\int \left(\frac{1}{x} - \frac{x}{1-x^2}\right)\,dx = -2\left(\ln|x| + \frac{1}{2}\ln|1-x^2|\right) + C.$$

Simplify:

$$\ln|v| = \ln\left|\frac{1}{x^2(1-x^2)}\right| + C, v = c_1\frac{1}{x^2(1-x^2)}.$$

Since $v = u'$,

$$u' = c_1\frac{1}{x^2(1-x^2)} = c_1\left(\frac{1}{x^2} + \frac{1}{2}\frac{1}{1+x} + \frac{1}{2}\frac{1}{1-x}\right),$$

by partial fractions.

Integrate to find u:

$$u = c_1\left(-\frac{1}{x} + \frac{1}{2}\ln|1+x| - \frac{1}{2}\ln|1-x|\right) + c_2.$$

Finally, solve for y:

$$y = ux = c_1\left(-1 + \frac{x}{2}\ln\left|\frac{1+x}{1-x}\right|\right) + c_2 x. \square$$

Problems on Reduction of Order.

1. Use reduction of order to solve $(D^2+1)y = \sec x$.

2. (a) Show that $y_1 = x^2$ is a solution to the **Euler equation**

$$x^2y'' + 2xy' - 6y = 0.$$

(b) Use reduction of order to find the general solution to

$$x^2y'' + 2xy' - 6y = 0.$$

Solutions to the Problems on Reduction of Order.

1. Use reduction of order to solve $(D^2+1)y = \sec x$.

$y_1 = \cos x$ is a homogeneous solution for $(D^2+1)y = \sec x$. (You could use $y_1 = \sin x$ as well.) Set $y = u\cos x$. Compute the derivatives:

$$y' = u'\cos x - u\sin x, \quad y'' = u''\cos x - 2u'\sin x - u\cos x.$$

Substitute:

$$\begin{array}{rcccccl} y''+y & = & u''\cos x & - & 2u'\sin x & - & u\cos x \\ & & & & & + & u\cos x \\ & = & u''\cos x & - & 2u'\sin x & & \quad = \sec x. \end{array}$$

Let $v = u'$, so $v' = u''$:

$$v'\cos x - 2v\sin x = \sec x, \quad v' - 2\frac{\sin x}{\cos x}v = (\sec x)^2.$$

This is first-order linear; the integrating factor is

$$I = \exp \int -2\frac{\sin x}{\cos x}\, dx = (\cos x)^2.$$

Therefore,

$$v(\cos x)^2 = \int dx = x + c_1, \quad v = x(\sec x)^2 + c_1(\sec x)^2.$$

$v = u'$, so

$$u' = x(\sec x)^2 + c_1(\sec x)^2, \quad u = x\tan x - \ln|\sec x| + c_1 \tan x + c_2.$$

(The work for the integral is shown below.)
Finally, $y = u\cos x$, so

$$y = u\cos x = x\sin x - (\cos x)\ln|\sec x| + c_1 \sin x + c_2 \cos x.$$

Here's the work for the integral I did to find u:

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \int dx \\ + & x & & (\sec x)^2 \\ & & \searrow & \\ - & 1 & & \tan x \\ & & \searrow & \\ + & 0 & \rightarrow & \ln|\sec x| \end{array}$$

$$\int x(\sec x)^2\, dx = x\tan x - \ln|\sec x| + C. \square$$

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2. (a) Show that $y_1 = x^2$ is a solution to the **Euler equation**

$$x^2y'' + 2xy' - 6y = 0.$$

All I have to do is plug $y_1 = x^2$ into the equation. Compute the derivatives:

$$y_1' = 2x, \quad y_1'' = 2.$$

Plug in and check that it works:

$$x^2y_1'' + 2xy_1 - 6y_1 = 2x^2 + 4x^2 - 6x^2 = 0. \square$$

(b) Use reduction of order to find the general solution to

$$x^2y'' + 2xy' - 6y = 0.$$

Set $y = ux^2$. Compute the derivatives:

$$y' = u'x^2 + 2ux, \quad y'' = u''x^2 + 4u'x + 2u.$$

Plug these into the original equation:

$$\begin{array}{rclcccl} x^2y'' + 2xy' - 6y & = & u''x^4 & + & 4u'x^3 & + & 2ux^2 \\ & & & + & 2u'x^3 & + & 4ux^2 \\ & & & & & - & 6ux^2 \\ & = & u''x^4 & + & 6u'x^3 & & = 0. \end{array}$$

Divide through by x^4:

$$u'' + \frac{6}{x}u' = 0.$$

Let $v = u'$, so $v' = u''$. Then

$$v' + \frac{6}{x}v = 0.$$

Separate variables:

$$\int \frac{dv}{v} = -6\int \frac{dx}{x}, \quad \ln|v| = -6\ln|x| + C, \quad v = \frac{c_0}{x^6}.$$

$v = u'$ so

$$u' = \frac{c_0}{x^6}, \quad u = \frac{c_1}{x^5} + c_2.$$

The solution is

$$y = ux^2 = \frac{c_1}{x^3} + c_2x^2. \ \square$$