4-18-1994

Review: The exponential of a matrix.

Consider a homogeneous linear system with constant coefficients:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \\ &\ldots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n. \end{aligned}$$

In matrix form, this is

$$\vec{x}\,' = A\vec{x},$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

and $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$.

This equation has the same form as the **exponential growth equation**

$$y' = ay,$$

whose solution is $y = ce^{at}$. It is natural to wonder whether the solution to the system might have the form

$$\vec{x} = e^{At}\vec{c},$$

where e^{At} would represent some sort of "matrix exponential".

In fact, it is possible to do this, and it provides a convenient method for solving constant coefficient linear systems.

To make sense of the matrix exponential e^{At}, let A be an $n \times n$ matrix. Define

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}.$$

I obtained this series by plugging At into the standard power series for the exponential function.

It is possible to show that this series converges absolutely for any $n \times n$ matrix A and any t. Hence, the definition makes sense, and you can also perform operations such as term-by-term differentiation.

Example. Let

$$A = \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} (-3)^2 & 0 \\ 0 & 4^2 \end{bmatrix}, \quad A^3 = \begin{bmatrix} (-3)^3 & 0 \\ 0 & 4^3 \end{bmatrix},$$

and in general,

$$A^n = \begin{bmatrix} (-3)^n & 0 \\ 0 & 4^n \end{bmatrix}.$$

Therefore,

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{bmatrix} (-3)^n & 0 \\ 0 & 4^n \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{t^n}{n!}(-3)^n & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{t^n}{n!}4^n \end{bmatrix} = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{4t} \end{bmatrix}.$$

It is easy to see that this will work for any **diagonal matrix** — i.e. a matrix whose only nonzero entries are on the main diagonal. Indeed, suppose that

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

is a diagonal matrix (so all the entries off the main diagonal are 0). Then

$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}. \ \square$$

Example. Let

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix},$$

and in general,

$$A^n = \begin{bmatrix} 1 & 0 \\ -2n & 1 \end{bmatrix}.$$

Hence,

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{bmatrix} 1 & 0 \\ -2n & 1 \end{bmatrix} = \begin{bmatrix} \sum_{n=0}^{\infty} \frac{t^n}{n!} & 0 \\ \sum_{n=0}^{\infty}(-2n)\frac{t^n}{n!} & \sum_{n=0}^{\infty} \frac{t^n}{n!} \end{bmatrix}.$$

Clearly,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} = e^t.$$

A little work is needed to evaluate the remaining sum:

$$\sum_{n=0}^{\infty}(-2n)\frac{t^n}{n!} = -2\sum_{n=1}^{\infty} \frac{t^n}{(n-1)!} = -2t\sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} = -2t\sum_{m=0}^{\infty} \frac{t^m}{m!} = -2te^t.$$

(I made the substitution $m = n - 1$ to get the third equality.)
Therefore,

$$e^{At} = \begin{bmatrix} e^t & 0 \\ -2te^t & e^t \end{bmatrix}. \ \square$$

Example. If you want e^A, simply set $t = 1$ in e^{At}. For instance, if

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix},$$

then (by the last example),

$$e^{At} = \begin{bmatrix} e^t & 0 \\ -2te^t & e^t \end{bmatrix}.$$

So

$$e^A = \begin{bmatrix} e & 0 \\ -2e & e \end{bmatrix}. \ \square$$

Example. It's possible to use the power series to show that the matrix exponential satisfies the usual properties of the exponential function:

$$e^{A+B} = e^A e^B \quad \text{provided that} \quad AB = BA,$$

$$\left(e^A\right)^{-1} = e^{-A}, \quad \text{and} \quad e^0 = I.$$

(In the last equation, 0 denotes the zero matrix, and I is the identity matrix.) $\square$

Next, I'll show how e^{At} solves the system

$$\vec{x}\,' = A\vec{x}.$$

Differentiate the power series for e^{At} term-by-term:

$$\frac{d}{dt}e^{At} = \frac{d}{dt}\sum_{n=0}^{\infty}\frac{A^n t^n}{n!} = \sum_{n=1}^{\infty} n\frac{A^n t^{n-1}}{n!} = \sum_{n=1}^{\infty}\frac{A^n t^{n-1}}{(n-1)!} =$$

$$A\sum_{n=1}^{\infty}\frac{A^{n-1}t^{n-1}}{(n-1)!} = A\sum_{m=0}^{\infty}\frac{A^m t^m}{m!} = Ae^{At}.$$

Therefore,

$$\frac{d}{dt}e^{At}\vec{c} = Ae^{At}\vec{c},$$

where $\vec{c} = \langle c_1, c_2, \ldots, c_n\rangle$. Thus, $\vec{x} = e^{At}\vec{c}$ is a solution to the system $\vec{x}\,' = A\vec{x}$. The existence and uniqueness theorem shows that this is *the* solution to the system.

Observe that setting $t = 0$ in $\vec{x} = e^{At}\vec{c}$ (and using the fact that $e^0 = I$) produces $\vec{x}(0) = \vec{c}$. In other words, $\vec{c}$ is the vectors of initial conditions at $t = 0$.

Since the exponential of a matrix yields the solution to a constant coefficient linear system, it would be nice to have a method for computing matrix exponentials. Direct computation using the power series (as in the examples above) will work on occasion, but is messy even in simple cases.

The following algorithm provides a convenient way of computing the exponential of a matrix. Suppose that A is an $n \times n$ matrix with eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. (Repeated eigenvalues must be repeated in the list according to their multiplicity.) Define functions

$$a_1(t) = e^{\lambda_1 t}, \quad a_k(t) = e^{\lambda_k t} \star a_{k-1}(t) = \int_0^t e^{\lambda_k(t-u)}a_{k-1}(u)\,du, \qquad 2 \le k \le n.$$

Defines matrices

$$B_1 = I, \quad B_k = B_{k-1}(A - \lambda_{k-1} I) = (A - \lambda_1 I)(A - \lambda_2 I) \ldots (A - \lambda_{k-1} I).$$

Then

$$e^{At} = a_1 B_1 + a_2 B_2 + \cdots + a_n B_n.$$

The proof that this formula works is straightforward; it uses an important result from linear algebra called the **Cayley-Hamilton theorem**. The idea is to verify that the expression above satisfies the equation

$$\frac{d}{dt}\vec{x} = A\vec{x}$$

just as e^{At} does.

Example. Use the matrix exponential to solve the linear system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \qquad x(0) = 1, y(0) = -3.$$

First, compute the eigenvalues. The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 2 \\ 3 & 2 - \lambda \end{vmatrix} = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1).$$

The eigenvalues are $\lambda = 4$ and $\lambda = -1$.
First, compute the a_k's:

$$a_1 = e^{4t}, \quad a_2 = e^{-t} \star e^{4t} = \int_0^t e^{-(t-u)} e^{4u}\, du = \frac{1}{5}e^{4t} - \frac{1}{5}e^{-t}.$$

Next, compute the B_k's:

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = A - 4I = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix}.$$

Therefore,

$$e^{At} = a_1 B_1 + a_2 B_2 = \begin{bmatrix} \frac{3}{5}e^{-t} + \frac{2}{5}e^{4t} & -\frac{2}{5}e^{-t} + \frac{2}{5}e^{4t} \\ -\frac{3}{5}e^{-t} + \frac{3}{5}e^{4t} & \frac{2}{5}e^{-t} + \frac{3}{5}e^{4t} \end{bmatrix}.$$

The general solution is

$$\vec{x} = \begin{bmatrix} \frac{3}{5}e^{-t} + \frac{2}{5}e^{4t} & -\frac{2}{5}e^{-t} + \frac{2}{5}e^{4t} \\ -\frac{3}{5}e^{-t} + \frac{3}{5}e^{4t} & \frac{2}{5}e^{-t} + \frac{3}{5}e^{4t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

The solution satisfying $\vec{x}(0) = \langle 1, -3 \rangle$ is

$$\vec{x} = \begin{bmatrix} \frac{3}{5}e^{-t} + \frac{2}{5}e^{4t} & -\frac{2}{5}e^{-t} + \frac{2}{5}e^{4t} \\ -\frac{3}{5}e^{-t} + \frac{3}{5}e^{4t} & \frac{2}{5}e^{-t} + \frac{3}{5}e^{4t} \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix}. \square$$

Problems on Matrix Exponentials.

1. If

$$e^{At} = \begin{bmatrix} -te^{5t} + e^{5t} & -te^{5t} \\ te^{5t} & te^{5t} + e^{5t} \end{bmatrix},$$

what is A?

2. Use the matrix exponential to find the general solution to

$$\vec{x}\,' = \begin{bmatrix} 1 & -3 \\ 3 & -5 \end{bmatrix} \vec{x}.$$

Describe the behavior of an arbitrary solution $\langle x(t), y(t)\rangle$ as $t \to \infty$.

3. Use the matrix exponential to find the solution to

$$\vec{x}\,' = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix} \vec{x}$$

which satisfies $\vec{x}(0) = \langle 2, 3\rangle$. Compute

$$\lim_{t\to\infty} x(t) \qquad \text{and} \qquad \lim_{t\to\infty} y(t)$$

for this solution.

4. Compute e^{At} for

$$A = \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix}.$$

Express the answer without using complex numbers or functions.

5. Use the matrix exponential to find the general solution to

$$\vec{x}\,' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -4 & 0 \\ 0 & -1 & 1 \end{bmatrix} \vec{x}.$$

Solutions to the Problems on Matrix Exponentials.

1. If

$$e^{At} = \begin{bmatrix} -te^{5t} + e^{5t} & -te^{5t} \\ te^{5t} & te^{5t} + e^{5t} \end{bmatrix},$$

what is A?

Observe that

$$\left.\frac{d}{dt}e^{At}\right|_{t=0} = Ae^{At}\big|_{t=0} = A \cdot I = A.$$

Therefore,

$$\frac{d}{dt}e^{At} = \begin{bmatrix} -5te^{5t} + 4e^{5t} & -5te^{5t} - e^{5t} \\ 5te^{5t} + e^{5t} & 5te^{5t} + 6e^{5t} \end{bmatrix},$$

$$A = \frac{d}{dt}e^{At}\bigg|_{t=0} = \begin{bmatrix} 4 & -1 \\ 1 & 6 \end{bmatrix}. \quad \square$$

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2. Use the matrix exponential to find the general solution to

$$\vec{x}\,' = \begin{bmatrix} 1 & -3 \\ 3 & -5 \end{bmatrix} \vec{x}.$$

Describe the behavior of an arbitrary solution $\langle x(t), y(t)\rangle$ as $t \to \infty$.

The characteristic polynomial is $\lambda^2 + 4\lambda + 4 = (\lambda + 2)^2$. The eigenvalue is $\lambda = -2$ (double). First, compute the a_k's:

$$a_1 = e^{-2t}, \quad a_2 = e^{-2t} \star e^{-2t} = \int_0^t e^{-2(t-u)}e^{-2u}\,du = ue^{-2u}.$$

Next, compute the B_k's:

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = A + 2I = \begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}.$$

Therefore,

$$e^{At} = \begin{bmatrix} e^{-2t} + 3te^{-2t} & -3te^{-2t} \\ 3te^{-2t} & e^{-2t} - 3te^{-2t} \end{bmatrix}.$$

The general solution is

$$\vec{x} = \begin{bmatrix} e^{-2t} + 3te^{-2t} & -3te^{-2t} \\ 3te^{-2t} & e^{-2t} - 3te^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Since $e^{-2t} \to 0$ and $te^{-2t} \to 0$ as $t \to \infty$, it follows that $\langle x(t), y(t)\rangle \to \vec{0}$ as $t \to \infty$. That is, the critical point at the origin is a stable solution which attracts every other solution. $\square$

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3. Use the matrix exponential to find the solution to

$$\vec{x}\,' = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix} \vec{x}$$

which satisfies $\vec{x}(0) = \langle 2, 3\rangle$. Compute

$$\lim_{t\to\infty} x(t) \qquad \text{and} \qquad \lim_{t\to\infty} y(t)$$

for this solution.

The characteristic polynomial is $\lambda^2 + \lambda = \lambda(\lambda + 1)$. The eigenvalues are $\lambda = 0$ and $\lambda = -1$. First, compute the a_k's:

$$a_1 = 1, \quad a_2 = e^{-t} \star 1 = \int_0^t e^{-(t-u)} \cdot 1\,du = 1 - e^{-t}.$$

Next, compute the B_k's:

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = A + 0 \cdot I = \begin{bmatrix} 2 & -6 \\ 1 & -3 \end{bmatrix}.$$

Therefore,

$$e^{At} = \begin{bmatrix} 3 - 2e^{-t} & -6 + 6e^{-t} \\ 1 - e^{-t} & -2 + 3e^{-t} \end{bmatrix}.$$

The solution satisfying $\vec{x}(0) = \langle 2, 3 \rangle$ is

$$\vec{x} = \begin{bmatrix} 3 - 2e^{-t} & -6 + 6e^{-t} \\ 1 - e^{-t} & -2 + 3e^{-t} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Multiply out the right side:

$$x = -12 + 14e^{-t}, \quad y = -4 + 7e^{-t}.$$

Since $e^{-t} \to 0$ as $t \to \infty$,

$$\lim_{t\to\infty} x(t) = -12 \qquad \text{and} \qquad \lim_{t\to\infty} y(t) = -4. \ \Box$$

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4. Compute e^{At} for

$$A = \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix}.$$

Express the answer without using complex numbers or functions.

The characteristic polynomial is $\lambda^2 + 4$. The eigenvalues are $\lambda = \pm 2i$.
First, compute the a_k's:

$$a_1 = e^{2it} = \cos 2t + i \sin 2t, \quad a_2 = e^{-2it} \star e^{2it} = \int_0^t e^{-2i(t-u)} e^{2iu}\, du = \frac{1}{4i}\left(e^{2it} - e^{-2it}\right) = \frac{1}{2}\sin 2t.$$

(I used the fact that $e^{2it} - e^{-2it} = (\cos 2t + i \sin 2t) - (\cos 2t - i \sin 2t) = 2i \sin 2t$.)
Next, compute the B_k's:

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = A - (2i)I = \begin{bmatrix} 2 - 2i & -4 \\ 2 & -2 - 2i \end{bmatrix}.$$

Therefore,

$$e^{At} = \begin{bmatrix} \cos 2t + i \sin 2t & 0 \\ 0 & \cos 2t + i \sin 2t \end{bmatrix} + \begin{bmatrix} \sin 2t - i \sin 2t & -2 \sin 2t \\ \sin 2t & -\sin 2t - i \sin 2t \end{bmatrix} =$$

$$\begin{bmatrix} \cos 2t + \sin 2t & -2 \sin 2t \\ \sin 2t & \cos 2t - \sin 2t \end{bmatrix}.$$

(Personally, I think computing e^{At} when the eigenvalues are complex can be a bit too messy. If all you're trying to do is solve a system, it might be better to use the eigenvector method.) $\Box$

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5. Use the matrix exponential to find the general solution to

$$\vec{x}\,' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -4 & 0 \\ 0 & -1 & 1 \end{bmatrix} \vec{x}.$$

Since the matrix is triangular, the diagonal entries are the eigenvalues: $\lambda = 1$ (double) and $\lambda = -4$.

First, compute the a_k's:

$$a_1 = e^t, \quad a_2 = e^t \star e^t = \int_0^t e^{t-u} e^u \, du = ue^u,$$

$$a_3 = e^{-4t} \star te^t = \int_0^t e^{-4(t-u)} ue^u \, du = \frac{1}{5}te^t - \frac{1}{25}e^t + \frac{1}{25}e^{-4t}.$$

Next, compute the B_k's:

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B_2 = A - I = \begin{bmatrix} 0 & 0 & 0 \\ 2 & -5 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \quad B_3 = (A - I)B_2 = \begin{bmatrix} 0 & 0 & 0 \\ -10 & 25 & 0 \\ -2 & 5 & 0 \end{bmatrix}.$$

Therefore,

$$e^{At} = \begin{bmatrix} e^t & 0 & 0 \\ \frac{2}{5}e^t - \frac{2}{5}e^{-4t} & e^{-4t} & 0 \\ -\frac{2}{5}te^t + \frac{2}{25}e^t - \frac{2}{25}e^{-4t} & -\frac{1}{5}e^t + \frac{1}{5}e^{-4t} & e^t \end{bmatrix}.$$

The general solution is

$$\vec{x} = \begin{bmatrix} e^t & 0 & 0 \\ \frac{2}{5}e^t - \frac{2}{5}e^{-4t} & e^{-4t} & 0 \\ -\frac{2}{5}te^t + \frac{2}{25}e^t - \frac{2}{25}e^{-4t} & -\frac{1}{5}e^t + \frac{1}{5}e^{-4t} & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \square$$