4-24-1994

Linear systems and eigenvectors.

A **constant coefficient homogeneous linear system** is a system of differential equations of the form

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n, \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n, \\ &\ldots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n. \end{aligned}$$

Rewrite the equation in matrix form:

$$\vec{x}\,' = A\vec{x}.$$

Here

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$$

and $\vec{x} = \langle x_1, x_2, \ldots, x_n \rangle$.

The **eigenvalues** and **eigenvectors** of A can be used to construct the solution to the system; they also provide information about the shape of the solution curves. I'll begin by reviewing how you compute the eigenvalues and eigenvectors of a matrix.

Example. Let

$$A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}.$$

The **characteristic polynomial** of A is

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & -4 \\ 0 & 5-\lambda & 4 \\ -4 & 4 & 3-\lambda \end{vmatrix} = -81 + 9\lambda + 9\lambda^2 - \lambda^3.$$

The **eigenvalues** (or **characteristic values**) of A are the roots of the characteristic polynomial. In this case, the eigenvalues are $\lambda = 3$, $\lambda = -3$, and $\lambda = 9$.

If λ is an eigenvalue for A, an **eigenvector** for λ is a *nonzero* vector $\vec{v}$ such that

$$(A - \lambda I)\vec{v} = \vec{0}.$$

You can find such a vector by solving the appropriate system of linear equations (e.g. by row-reduction). For example, consider the eigenvalue $\lambda = 3$.

$$A - 3I = \begin{bmatrix} -2 & 0 & -4 \\ 0 & 2 & 4 \\ -4 & 4 & 0 \end{bmatrix}.$$

I want a vector $\langle a, b, c \rangle$ such that

$$\begin{bmatrix} -2 & 0 & -4 \\ 0 & 2 & 4 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

You can solve this system of equations by row reducing the matrix

$$\begin{bmatrix} -2 & 0 & -4 & 0 \\ 0 & 2 & 4 & 0 \\ -4 & 4 & 0 & 0 \end{bmatrix}.$$

However, the last column will always consist of zeros. Therefore, I may omit it to save writing. (Naturally, I have to put it back when the row-reduction is complete.)

$$\begin{bmatrix} -2 & 0 & -4 \\ 0 & 2 & 4 \\ -4 & 4 & 0 \end{bmatrix} \underset{r_1 \to -r_1/2}{\to} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ -4 & 4 & 0 \end{bmatrix} \underset{r_3 \to r_3 + 4r_1}{\to} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 4 & 8 \end{bmatrix}$$

$$\underset{r_3 \to r_3 - 2r_2}{\to} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix} \underset{r_2 \to r_2/2}{\to} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Put the column of zeros back and write down the corresponding equations:

$$\begin{array}{rcrcrcl} a & & & + & 2c & = & 0 \\ & & b & + & 2c & = & 0 \end{array}$$

This means that $a = -2c$ and $b = -2c$, so

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2c \\ -2c \\ c \end{bmatrix} = c \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}.$$

I may choose c to be any nonzero number. (In these situations, it's nice to choose a number which "clears" any fractions.) In this case, I can take $c = 1$, so the eigenvector is $\langle -2, -2, 1 \rangle$.

In the future, I'll follow this procedure for finding eigenvectors without comment. Here are eigenvectors for the other two eigenvalues.

Consider $\lambda = -3$:

$$A + 3I = \begin{bmatrix} 4 & 0 & -4 \\ 0 & 8 & 4 \\ -4 & 4 & 6 \end{bmatrix} \underset{r_1 \to r_1/4}{\to} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 8 & 4 \\ -4 & 4 & 6 \end{bmatrix} \underset{r_3 \to r_3 + 4r_1}{\to} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 8 & 4 \\ 0 & 4 & 2 \end{bmatrix}$$

$$\underset{r_2 \to r_2/8}{\to} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 4 & 2 \end{bmatrix} \underset{r_3 \to r_3 - 4r_2}{\to} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix says that $a - c = 0$ and $b + \frac{1}{2}c = 0$. Therefore, $a = c$ and $b = -\frac{1}{2}c$, and

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c \\ -\frac{1}{2}c \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}.$$

Take $c = 2$ to obtain the eigenvector $\langle 2, -1, 2 \rangle$.

Finally, consider $\lambda = 9$:

$$A - 9I = \begin{bmatrix} -8 & 0 & -4 \\ 0 & -4 & 4 \\ -4 & 4 & -6 \end{bmatrix} \underset{r_1 \to -r_1/8}{\to} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & -4 & 4 \\ -4 & 4 & -6 \end{bmatrix} \underset{r_3 \to r_3 + 4r_1}{\to} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & -4 & 4 \\ 0 & 4 & -4 \end{bmatrix}$$

$$\underset{r_3 \to r_3 + r_2}{\to} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix} \underset{r_2 \to -r_2/4}{\to} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix says that $a + \frac{1}{2}c = 0$ and $b - c = 0$. Therefore, $a = -\frac{1}{2}c$ and $b = c$, and

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}c \\ c \\ c \end{bmatrix} = c \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 1 \end{bmatrix}.$$

Take $c = 2$ to obtain the eigenvector $\langle -1, 2, 2 \rangle$.

An $n \times n$ matrix A with n independent eigenvectors is **diagonalizable**. In this case, the matrix is 3×3, and there are 3 independent eigenvectors, so the matrix is diagonalizable.

To understand the terminology, construct a matrix S by using the eigenvectors as the columns:

$$S = \begin{bmatrix} -2 & 2 & -1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix}.$$

Find the **inverse** of S (e.g. by row-reduction):

$$S^{-1} = \begin{bmatrix} -\frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \\ \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{9} & \frac{2}{9} & \frac{2}{9} \end{bmatrix}.$$

Then $S^{-1} A S$ will be a **diagonal matrix**, i.e. a matrix whose only nonzero entries are on the main diagonal. The diagonal entries are the eigenvalues of A, listed in the same order as the corresponding eigenvectors used to construct S.

$$S^{-1} A S = \begin{bmatrix} -\frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \\ \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{9} & \frac{2}{9} & \frac{2}{9} \end{bmatrix} \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} \begin{bmatrix} -2 & 2 & -1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 9 \end{bmatrix}. \quad \square$$

Example. Let

$$A = \begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix}.$$

The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -1 \\ 5 & -1 - \lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2.$$

The eigenvalues are $\lambda = 1 \pm i$. I'll begin by finding an eigenvector for $\lambda = 1 + i$.

$$A - (1 + i)I = \begin{bmatrix} 2 - i & -1 \\ 5 & -2 - i \end{bmatrix}.$$

Here is a useful shortcut. *It only applies to 2×2 matrices of the form $A - \lambda I$ that come up in eigenvector computations.*

In such a matrix, the two rows are multiples of one another. This means that I can kill one of the rows by subtracting a multiple of the other row. I don't actually have to perform the row operation; I can just "zero out" one of the rows:

$$\begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix} \to \begin{bmatrix} 2-i & -1 \\ 0 & 0 \end{bmatrix}$$

Here is another shortcut. I don't have to go all the way to row reduced echelon form if I keep in mind what I want to do. I'm trying to find a nonzero vector $\langle a, b \rangle$ such that

$$\begin{bmatrix} 2-i & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This means that

$$(2-i) \cdot a + (-1) \cdot b = 0.$$

There is an obvious way to get a solution — just "swap" the numbers $2-i$ and -1 and negate one of them (say -1). This will work, because

$$(2-i) \cdot (1) + (-1) \cdot (2-i) = 0.$$

Thus, $a = 1$, $b = 2 - i$ is a solution, and $\langle 1, 2-i \rangle$ is an eigenvector.

(I don't need to go all the way to row reduced echelon form because I don't need the general solution — any particular solution will do.)

Now I want an eigenvector for $\lambda = 1 - i$. Here is yet another shortcut. Suppose $c \pm di$ is a pair of complex conjugate eigenvalues, and $\vec{v}$ is an eigenvector for one of the eigenvalues. Then the conjugate $\vec{v}^\star$ is an eigenvector for the other eigenvalue.

(You find the conjugate of a complex vector by taking the conjugate of each component.)

In this example, since $\langle 1, 2-i \rangle$ is an eigenvector for $1 + i$, it follows that $\langle 1, 2+i$ is an eigenvector for $1 - i$. $\square$

Suppose the $n \times n$ matrix A has n independent eigenvectors. I will show you two ways to write down the solution to the system

$$\vec{x}\,' = A\vec{x}.$$

The first approach makes use of the fact that the matrix is diagonalizable. Suppose the n independent eigenvectors are $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, and the corresponding eigenvalues are $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. (Note: A given eigenvalue may give rise to several independent eigenvectors. In that case, the eigenvalue is repeated a corresponding number of times in the list.)

Let S be the matrix whose columns are the eigenvectors (as in the preceding example). Thus,

$$S = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}.$$

The general solution to the system is

$$\vec{x} = S \begin{bmatrix} e^{\lambda_1 1} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} S^{-1}.$$

(The matrix SAS^{-1} that appears in the solution is actually the **matrix exponential** e^{At}.)

Example. Consider the system

$$\vec{x}\,' = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} \vec{x}.$$

Thus,

$$A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}.$$

By an earlier example, the eigenvalues of A are $\lambda = 3$, $\lambda = -3$, and $\lambda = 9$. The corresponding eigenvectors are $\langle -2, -2, 1 \rangle$, $\langle 2, -1, 2 \rangle$, and $\langle -1, 2, 2 \rangle$.

The matrix S whose columns are the eigenvectors is

$$S = \begin{bmatrix} -2 & 2 & -1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix}.$$

The inverse of S is:

$$S^{-1} = \begin{bmatrix} -\frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \\ \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{9} & \frac{2}{9} & \frac{2}{9} \end{bmatrix}.$$

Therefore, the solution to the system is

$$\vec{x} = \begin{bmatrix} -2 & 2 & -1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & e^{-3t} & 0 \\ 0 & 0 & e^{9t} \end{bmatrix} \begin{bmatrix} -\frac{2}{9} & -\frac{2}{9} & \frac{1}{9} \\ \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{1}{9} & \frac{2}{9} & \frac{2}{9} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \quad \Box$$

Alternatively, you can write the solution as a linear combination of fundamental solutions. Each eigenvalue λ_k with eigenvector $\vec{v}_k$ gives rise to a fundamental solution $e^{\lambda_k t}\vec{v}_k$. The general solution is the sum of these fundamental solutions, each multiplied by an arbitrary constant.

Example. The system

$$\vec{x}\,' = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix} \vec{x}$$

has eigenvalues $\lambda = 3$, $\lambda = -3$, $\lambda = 9$ and (corresponding) eigenvectors $\langle -2, -2, 1 \rangle$, $\langle 2, -1, 2 \rangle$, $\langle -1, 2, 2 \rangle$.

The solution is

$$\vec{x} = c_1 e^{3t} \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + c_3 e^{9t} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}. \quad \Box$$

Example. Real eigenvectors appear in the phase portraits of linear systems as linear solutions. Consider the system

$$\vec{x}\,' = \begin{bmatrix} -9 & -4 \\ 9 & 3 \end{bmatrix} \vec{x}.$$

The coefficient matrix is

$$A = \begin{bmatrix} -9 & -4 \\ 9 & 3 \end{bmatrix}.$$

The characteristic polynomial is $9+6\lambda+\lambda^2$. The eigenvalue is $\lambda = -3$ (double). Here is the eigenvector computation:

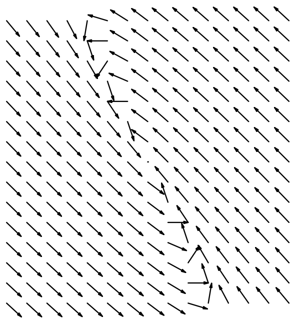
$$A + 3I = \begin{bmatrix} -6 & -4 \\ 9 & 6 \end{bmatrix} \underset{r_1 \to -r_1/6}{\longrightarrow} \begin{bmatrix} 1 & \frac{2}{3} \\ 9 & 6 \end{bmatrix} \underset{r_2 \to r_2 - 9r_1}{\longrightarrow} \begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 0 \end{bmatrix}$$

The last matrix says that $a + \frac{2}{3}b = 0$, so $a = -\frac{2}{3}b$. Therefore,

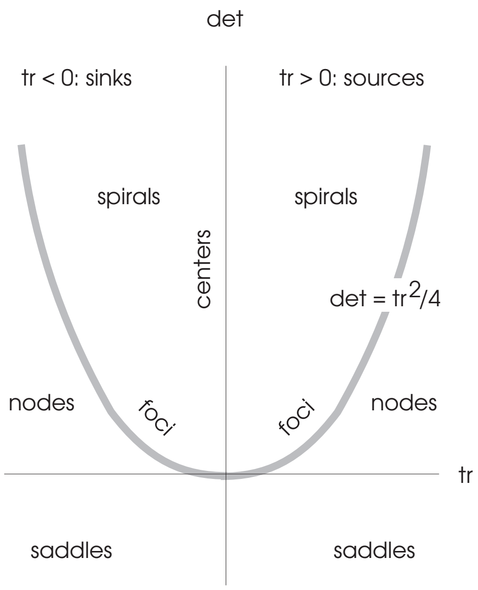
$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}b \\ b \end{bmatrix} = b \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}.$$

Setting $b = 3$ yields the eigenvector $\langle -2, 3 \rangle$.

Now consider the phase portrait for the system:



Phase portraits for 2-dimensional linear systems are classified by computing the determinant and the trace of the coefficient matrix.



In this example, the determinant of A is 9 and the trace of A is -6. Since $\det = \dfrac{1}{4}\operatorname{tr}^2$, the phase portrait (or the critical point $(0,0)$) is an **(improper) node**. Moreover, since the trace is negative, the origin is **stable**: it attracts the other solution curves.

Observe that there is a pair of linear solutions lying on the line which stretches through the picture from northwest to southeast. The line is determined by the eigenvector $\langle -2, 3\rangle$, which has slope $-\dfrac{3}{2}$.

Thus, if the system has real eigenvectors, they appear on the phase portrait as linear solutions passing through the origin.

A final remark: Since this system does not have two independent eigenvectors, you cannot use the methods described earlier to write down the solution to the system. There are several ways to obtain the solution in this case. You can use "brute force", reducing the two equations so a single equation involving x or y. You can use a variation on the eigenvector method which is similar to our handling of repeated roots for linear constant coefficient differential equations. Finally, you can find the solution using the **matrix exponential**. $\square$

Example. Solve

$$\vec{x}\,' = \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix} \vec{x}.$$

The characteristic polynomial is $4 + \lambda^2$. The eigenvalues are $\lambda = \pm 2i$. I'll use $\lambda = 2i$.

$$A - 2i \cdot I = \begin{bmatrix} 2 - 2i & -4 \\ 2 & -2 - 2i \end{bmatrix}.$$

I want $\langle a, b\rangle$ such that

$$\begin{bmatrix} 2 - 2i & -4 \\ 2 & -2 - 2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The second row of $A - 2i \cdot I$ must be a multiple of the first, so kill it:

$$\begin{bmatrix} 2 - 2i & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Now I want to solve

$$(2 - 2i) \cdot a + (-4) \cdot b = 0.$$

An obvious solution is $a = 4$, $b = 2 - 2i$. The eigenvector is $\langle 4, 2 - 2i\rangle$. It doesn't hurt to multiply an eigenvector by a number, so I'll factor out 2 and use $\langle 2, 1 - i\rangle$ instead.

The solution generated by this eigenvector is

$$e^{2it} \begin{bmatrix} 2 \\ 1 - i \end{bmatrix}.$$

I can obtain two independent solutions for the original equation by taking the real and imaginary parts of this solution. Expand it:

$$e^{2it} \begin{bmatrix} 2 \\ 1 - i \end{bmatrix} = (\cos 2t + i \sin 2t) \begin{bmatrix} 2 \\ 1 - i \end{bmatrix} = \begin{bmatrix} 2\cos 2t & + & 2\sin 2t \\ (\cos 2t + \sin 2t) & + & i(-\cos 2t + \sin 2t) \end{bmatrix}.$$

The real and imaginary parts are

$$\operatorname{re} e^{2it} \begin{bmatrix} 2 \\ 1 - i \end{bmatrix} = \begin{bmatrix} 2\cos 2t \\ \cos 2t + \sin 2t \end{bmatrix},$$

$$\operatorname{im} e^{2it} \begin{bmatrix} 2 \\ 1 - i \end{bmatrix} = \begin{bmatrix} 2\sin 2t \\ -\cos 2t + \sin 2t \end{bmatrix}.$$

Therefore, the general solution is

$$\vec{x} = c_1 \begin{bmatrix} 2\cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + c_2 \begin{bmatrix} 2\sin 2t \\ -\cos 2t + \sin 2t \end{bmatrix}. \quad \square$$

Problems on Linear systems and eigenvectors.

1. Let

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}.$$

(a) Find the eigenvalues of A, and the independent eigenvectors corresponding to each eigenvalue. Find a matrix S such that $S^{-1}AS$ is diagonal.

(b) Find the general solution to the linear system

$$\vec{x}\,' = A\vec{x}.$$

2. Consider the linear system

$$\vec{x}\,' = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \vec{x}.$$

(a) Find the eigenvectors for the coefficient matrix, and use them to write down a solution for the system.

(b) Classify the phase portrait of the system by computing the determinant and the trace.

(c) For a solution satisfying $x(5) = 4$, $y(5) = 0$, what is $\lim_{t\to\infty} \dfrac{y}{x}$?

3. Find the general solution to the system

$$\vec{x}\,' = \begin{bmatrix} -1 & 5 \\ -1 & -3 \end{bmatrix} \vec{x}.$$

(Express your answer without using complex numbers or functions.)

Solutions to the Problems on Linear systems and eigenvectors.

1. Let

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}.$$

(a) Find the eigenvalues of A, and the independent eigenvectors corresponding to each eigenvalue. Find a matrix S such that $S^{-1}AS$ is diagonal.

The characteristic polynomial of A is $-\lambda + 2\lambda^2 - \lambda^3$. The eigenvalues are $\lambda = 0$ and $\lambda = 1$ (double). Consider $\lambda = 0$:

$$A - 0 \cdot I = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \underset{r_1 \to r_1 - 2r_2}{\longrightarrow} \begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \underset{r_3 \to r_3 - r_2}{\longrightarrow} \begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\underset{r_1 \leftrightarrow r_2}{\rightarrow} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \underset{r_1 \to r_1 + 2r_2}{\rightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \underset{r_3 \to r_3 + r_2}{\rightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix says that $a - c = 0$ and $b - c = 0$. Therefore, $a = c$ and $b = c$, and

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Take $c = 1$ to obtain the eigenvector $\langle 1, 1, 1 \rangle$.
Now consider $\lambda = 1$:

$$A - I = \begin{bmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 1 & -3 & 1 \end{bmatrix} \underset{r_2 \to r_2 - r_1}{\rightarrow} \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 1 & -3 & 1 \end{bmatrix} \underset{r_3 \to r_3 - r_1}{\rightarrow} \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix says that $a - 3b + c = 0$. Therefore,

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3b - c \\ b \\ c \end{bmatrix} = b \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

The eigenvectors are $\langle 3, 1, 0 \rangle$ and $\langle -1, 0, 1 \rangle$.
Now

$$S = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad S^{-1} = \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}.$$

Therefore,

$$S^{-1}\, A\, S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad \square$$

(b) Find the general solution to the linear system

$$\vec{x}\,' = A\vec{x}.$$

$$\vec{x} = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} -1 & 3 & -1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

If I hadn't found S and S^{-1} in the first part of the problem, it would have been more efficient to use the eigenvectors to write the solution:

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_3 e^t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \quad \square$$

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2. Consider the linear system

$$\vec{x}\,' = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \vec{x}.$$

(a) Find the eigenvectors for the coefficient matrix, and use them to write down a solution for the system.

The characteristic polynomial is $-6 - \lambda + \lambda^2$. The eigenvalues are $\lambda = -2$ and $\lambda = 3$.
Consider $\lambda = -2$:

$$A + 2I = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

Throw out the first row. I want a and b such that

$$2 \cdot a + 1 \cdot b = 0.$$

An obvious solution is $\langle a, b\rangle = \langle 1, -2\rangle$.
Next, consider $\lambda = 3$:

$$A - 3I = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$$

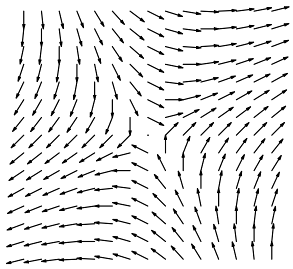
Throw out the second row. I want a and b such that

$$1 \cdot a + (-2) \cdot b = 0.$$

An obvious solution is $\langle a, b\rangle = \langle 2, 1\rangle$.
The solution is

$$\vec{x} = c_1 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \ \square$$

(b) Classify the phase portrait of the system by computing the determinant and the trace.



Since the determinant is -6, the critical point $(0, 0)$ is a **saddle**. $\square$

(c) For a solution satisfying $x(5) = 4$, $y(5) = 0$, what is $\lim_{t\to\infty} \dfrac{y}{x}$?

The eigenvectors determine lines of slope $\dfrac{1}{2}$ and slope -2. The initial conditions $x(5) = 4$, $y(5) = 0$ refer to a solution crossing the positive x-axis at $(4, 0)$; since $y' = 2x - y$, we have $y' = 8$. Therefore, the solution curve is moving upward, and will approach the line $y = \dfrac{1}{2}x$ asymptotically as $t \to \infty$. For this line, $\dfrac{y}{x} = \dfrac{1}{2}$, so

$$\lim_{t\to\infty} \frac{y}{x} = \frac{1}{2}. \ \square$$

3. Find the general solution to the system

$$\vec{x}\,' = \begin{bmatrix} -1 & 5 \\ -1 & -3 \end{bmatrix} \vec{x}.$$

(Express your answer without using complex numbers or functions.)

The characteristic polynomial is $8 + 4\lambda + \lambda^2$. The eigenvalues are $\lambda = -2 \pm 2i$. I'll use $\lambda = -2 + 2i$.

$$A - (-2+2i)I = \begin{bmatrix} 1-2i & 5 \\ -1 & -1-2i \end{bmatrix}$$

I want a vector $\langle a, b\rangle$ such that

$$\begin{bmatrix} 1-2i & 5 \\ -1 & -1-2i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The two rows of $A - (-2+2i)I$ must be multiples, so throw out the second row. I want a and b such that

$$(1-2i) \cdot a + 5 \cdot b = 0.$$

An obvious solution is $a = -5$, $b = 1 - 2i$. The eigenvector is $\langle -5, 1-2i\rangle$.
Write down the corresponding solution and expand:

$$e^{(-2+2i)t} \begin{bmatrix} -5 \\ 1-2i \end{bmatrix} = e^{-2t} e^{2it} \begin{bmatrix} -5 \\ 1-2i \end{bmatrix} = e^{-2t}(\cos 2t + i \sin 2t) \begin{bmatrix} -5 \\ 1-2i \end{bmatrix} =$$

$$e^{-2t} \begin{bmatrix} -5\cos 2t & - & 5i \sin 2t \\ (\cos 2t + 2\sin 2t) & + & i(-2\cos 2t + \sin 2t) \end{bmatrix}$$

Take the real and imaginary parts:

$$\operatorname{re} e^{(-2+2i)t} \begin{bmatrix} -5 \\ 1-2i \end{bmatrix} = e^{-2t} \begin{bmatrix} -5\cos 2t \\ \cos 2t + 2\sin 2t \end{bmatrix},$$

$$\operatorname{im} e^{(-2+2i)t} \begin{bmatrix} -5 \\ 1-2i \end{bmatrix} = e^{-2t} \begin{bmatrix} -5\sin 2t \\ -2\cos 2t + \sin 2t \end{bmatrix}.$$

The general solution is

$$\vec{x} = c_1 e^{-2t} \begin{bmatrix} -5\cos 2t \\ \cos 2t + 2\sin 2t \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -5\sin 2t \\ -2\cos 2t + \sin 2t \end{bmatrix}. \quad \square$$