9-14-1993

Limits.

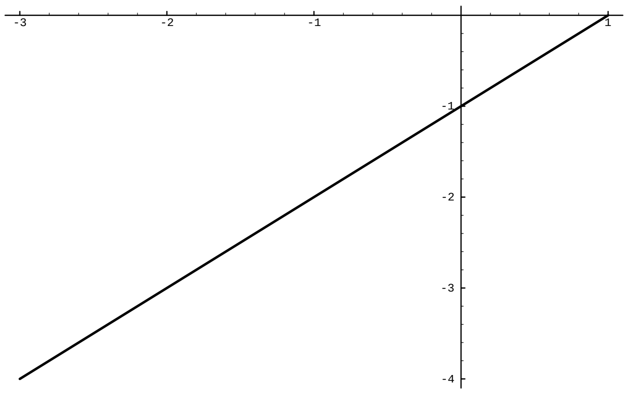
Calculus was used long before it was established on firm mathematical foundations. **Limits** give us a precise way of talking about **convergence** and infinite processes.

For example, we will define **derivatives** and **integrals** using limits. We will also use limits to study graphs.

Intuitively, we speak of **convergence** means that a variable quantity approaches a fixed number. For example, consider $f(x) = \dfrac{x^2-1}{x+1}$. Plug in numbers close to -1:

x	$f(x)$
-0.9	-1.9
-1.01	-2.01
-1.0001	-2.0001

It seems as though the $f(x)$-values are hovering around -2. If we graph $\dfrac{x^2-1}{x+1}$, the picture seems to confirm our guess:



Observe that $\dfrac{x^2-1}{x+1}$ is not defined at $x=-1$. *We are only interested in what it's approaching, not what it does when it gets there.*

(Do you understand *why* $\dfrac{x^2-1}{x+1}$ isn't defined at $x=-1$? If you plug $x=-1$ in, you get $\dfrac{0}{0}$. This is *not* equal to 1 — it is an **indeterminate form** meaning that it may have a value or it may be undefined. And no — you can't cancel the $x+1$, even though $x^2-1=(x-1)(x+1)$. Cancellation *assumes* that the thing you're cancelling is nonzero ... and of course, $x+1$ *is* 0 if $x=-1$.)

Notice that we are "bouncing around" -1, as opposed to plugging in a **sequence** of numbers "leading to" -1. Of course, we want to say that $\dfrac{x^2-1}{x+1}$ "goes to" -2 as x "goes to" -1 — but what does "goes to" mean?

To say that

$$\lim_{x\to a} f(x) = L$$

means that we can make $f(x)$ arbitrarily close to L by making x sufficiently close to a.

The last statement may seem rather abstract. Think of it as a *challenge*. You are trying to prove that $\lim_{x\to a} f(x) = L$ to Your Worst Enemy (YWE). YWE will challenge you like this: "How close does x have to be to a in order to guarantee that $f(x)$ is within ... 0.0001 of L?"

Since this is your *worst* enemy, you expect that (s)he will challenge you with a *very small* number. But it is worse than that!

YWE is also incredibly stubborn, and will continue challenging you indefinitely. Since you are a virtuous soul who would not think of resorting to violence, the only way you can beat YWE is to *prove* that you could answer the challenge *no matter what the number is.* You have to come up with a *rule* for answering the challenge for any number.

Here's an example. You say that

$$\lim_{x\to 2}(3x+1) = 7.$$

YWE challenges you to prove it. "How close does x have to be to 2 to make $3x+1$ stay within 0.1 of 7?" "How close does x have to be to 2 to make $3x+1$ stay within 0.01 of 7?" "How close does x have to be to 2 to make $3x+1$ stay within 0.001 of 7?" ...

You're *not* going to stand there answering one challenge after another — there's no end! Instead, you say to YWE: "If you challenge me with the number JUNK, my answer is going to be $\dfrac{1}{3}$JUNK."

Wow! Is that *tough*, or what?

Well, YWE doesn't give up so easily. (S)he says: "Prove to me that your rule will work, no matter what my number JUNK is."

You say: "To say x is within $\dfrac{1}{3}$JUNK of 2 means

$$|x-2| < \frac{1}{3}\text{JUNK."}$$

"By algebra, this gives

$$\begin{aligned} 3|x-2| &< \text{JUNK}, \\ |3x-6| &< \text{JUNK}, \\ |(3x+1)-7| &< \text{JUNK."} \end{aligned}$$

"The last line says that $3x+1$ is within JUNK — your number — of 7, which proves my claim."

YWE retires in disgrace.

Rather than *prove* all our manipulations in this way, we will fall back on certain rules for computing limits. The rules, of course, *are* proved using careful procedures like the one above.

Most of the rules are common sense. I won't list them all; here are a few important ones:

1. If $f(x)$ is a polynomial, then

$$\lim_{x\to c} f(x) = f(c).$$

Remember that a polynomial is a function of the form

$$a_0 + a_1x + \cdots + a_nx^n,$$

where the a's are numbers. The rule says you can compute the limit of a polynomial as x goes to c by plugging c in for x.

Example.
$$\lim_{x\to 1}(3x^4 - 2x + 17) = 3 - 2 + 17 = 18. \square$$

2. "The limit of a sum is the sum of the limits."

$$\lim_{x\to a}\left(f(x)+g(x)\right) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x).$$

Example. If I tell you that

$$\lim_{x\to 4} f(x) = 11,$$

then

$$\lim_{x \to 4} \left(f(x) + x^3\right) = 11 + 64 = 75. \ \square$$

There are a number of obvious-sounding rules that work like this. Let's list them without too much discussion.

3. "The limit of a product is the product of the limits."

$$\lim_{x \to a} (f(x) \cdot g(x)) = \left(\lim_{x \to a} f(x)\right) \cdot \left(\lim_{x \to a} g(x)\right).$$

We have to be a little careful with quotients to avoid stupid stuff.

4. "The limit of a quotient is the quotient of the limits" — provided that the denominator does not become 0 when you plug in. (If it does, you have some work to do — see below.)

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \text{ if } g(a) \neq 0.$$

5. "Constants pull out of limits." If k is a number, then

$$\lim_{x \to a} k \cdot f(x) = k \cdot \lim_{x \to a} f(x).$$

There are also rules for radicals, rational functions, and so on.
Here is a crude way to summarize all of these rules.

To compute $\lim_{x \to a} f(x)$, begin by plugging a in for x. If "nothing bad happens", then what you get is the answer.

Here is an example of our crude rule.

Example.

$$\lim_{x \to 1} \sqrt[3]{\frac{x^2 - 4}{x - 2}} = \sqrt[3]{\frac{-3}{-1}} = \sqrt[3]{3}. \ \square$$

Obviously, we need to discuss what it means for "something bad to happen". We will illustrate with some examples.

Examples.

1.

$$\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$$

If we plug -1 into $\dfrac{x^2 - 1}{x + 1}$, we get $\dfrac{0}{0}$. This is a Bad Thing — as we noted earlier, this is called an **indeterminate form**. As the name implies, *anything* can happen — the limit may be undefined, it may come out to a number, the Tribe might win the pennant. *Anything.*

What this means for you is that you have to do more work before you can come to a conclusion. Fortunately, this is easy.

We mentioned above that we don't care what happens *at* $x = -1$ — we only care about what $\dfrac{x^2-1}{x+1}$ is approaching. We may therefore take $x \neq -1$, in which case $x + 1 \neq 0$. Then

$$\lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1} \frac{(x-1)(x+1)}{x+1} = \lim_{x\to -1}(x-1),$$

where we may cancel the $x + 1$'s *because* $x + 1 \neq 0$.

Finally,

$$\lim_{x\to -1}(x-1) = -2$$

by plugging in — nothing bad happened this time. Notice that -2 was what we seemed to be getting above. □

We won't always describe the action in such excruciating detail, but you should understand *why* the algebraic manipulations we perform are legitimate. They usually reduce to the idea in the last example.

2.
$$\lim_{x\to 3} \frac{x^3-27}{x-3}$$

If we plug $x = 3$ into $\dfrac{x^3-27}{x-3}$, we get $\dfrac{0}{0}$. This means we have some work to do.

Since

$$x^3 - 27 = (x-3)(x^2+3x+9),$$

we have

$$\lim_{x\to 3} \frac{x^3-27}{x-3} = \lim_{x\to 3} \frac{(x-3)(x^2+3x+9)}{x-3} = \lim_{x\to 3}(x^2+3x+9) = 27. \;\square$$

Remark. In case you've forgotten,

$$a^3 - b^3 = (a-b)(a^2+ab+b^2), \text{ and } a^3+b^3 = (a+b)(a^3-ab+b^2).$$

But really, you don't need to memorize these formulas. For example, in the example above, you saw that plugging $x = 3$ into $x^3 - 27$ gives 0. This means that $x - 3$ must divide $x^3 - 27$; you can find the quotient $x^2 + 3x + 9$ by long division. □

3.
$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$$

If we plug $x = 4$ into $\dfrac{x-4}{\sqrt{x}-2}$, we get $\dfrac{0}{0}$. This means we have some work to do.

We have

$$x - 4 = (\sqrt{x}-2)(\sqrt{x}+2).$$

Therefore,

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{\sqrt{x}-2} = \lim_{x\to 4}(\sqrt{x}+2) = 4. \;\square$$

4.
$$\lim_{x\to 2} \frac{x^2+x-2}{x-2}$$

Plugging in gives $\frac{4}{0}$. The limit is *undefined*. In general, if you plug in and get

$$\frac{\text{nonzero number}}{0},$$

the limit is undefined. □

5. In some cases, we let x approach the number a from the left or the right, rather than "both sides at once" as usual.

In some cases, a one-sided limit is a logical necessity. For example, it wouldn't make sense to write $\lim_{x\to 0}\sqrt{x}$, because $\sqrt{x}$ is only defined for $x \ge 0$ (i.e. to the right of 0).

In other situations, the function may be defined on both sides of the limit point, but you may want to proceed by taking the limits from the left and right separately. For example, if

$$f(x) = \begin{cases} x^2 & \text{if } x \le 0 \\ x+1 & \text{if } x > 0 \end{cases}$$

the two pieces of the function are glued together at $x = 0$. It makes sense that we might want to compute $\lim_{x\to 0} f(x)$ by considering the left and right sides one at a time.

- $\lim_{x\to a+} f(x)$ means: Compute the limit of $f(x)$ as x approaches a from the *right*.
- $\lim_{x\to a-} f(x)$ means: Compute the limit of $f(x)$ as x approaches a from the *left*.

The left- and right-hand limits are the same exactly when the ordinary limit exists. In this case, the left-hand, right-hand, and ordinary limit are equal.

Consider the function $f(x)$ we defined above. If we approach 0 from the right, $x > 0$ and $f(x) = x + 1$. So

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+}(x+1) = 1.$$

On the other hand,

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x^2 = 0,$$

because $x \to 0-$ (x approaches 0 from the left) means $x < 0$, and so $f(x) = x^2$.

(Notice, by the way, that all the rules for computing limits that we discussed earlier apply to one-sided limits.)

The right-hand limit is 1; the left-hand limit is 0. Since the left- and right-hand limits do not agree,

$$\lim_{x\to 0} f(x) \text{ is undefined.}$$

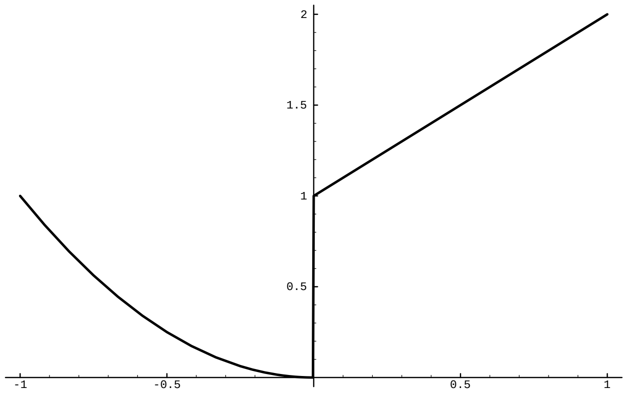
We can use *Mathematica* to graph the function. Since it's defined "in pieces", we define it before plotting it:

```
f[x_] := x^2 /; x <= 0
f[x_] := x + 1 /; x > 0
```

Now we can plot it:

```
Plot[f[x], {x, -1, 1}]
```

Here's the picture:



Some care is needed in dealing with absolute values. For example, let

$$g(x) = \begin{cases} \dfrac{x}{|x|} & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}$$

Does $\lim_{x \to 0} g(x)$ exist?

The right-hand limit is easy:

$$\lim_{x \to 0+} g(x) = \lim_{x \to 0+} (x - 1) = -1.$$

To compute $\lim_{x \to 0-} g(x)$, remember that approaching 0 from the left means that $x < 0$. Since $x < 0$, $|x| = -x$. Therefore,

$$\lim_{x \to 0-} g(x) = \lim_{x \to 0+} \frac{x}{|x|} = \lim_{x \to 0+} \frac{x}{-x} = \lim_{x \to 0+} -1 = -1.$$

Since the left- and right-hand limits agree, $\lim_{x \to 0} g(x)$ is defined, and

$$\lim_{x \to 0} g(x) = -1. \square$$

6.
$$\lim_{x \to 2+} \frac{x^2 + x - 2}{x - 2}$$

Plugging in gives $\dfrac{4}{0}$. The limit is *undefined.* But this time, we can do a little better.

Try plugging in a number a little bigger than 2: When $x = 2.001$, $\dfrac{x^2 + x - 2}{x - 2} \approx 4005$. It looks as though $\dfrac{x^2 + x - 2}{x - 2}$ is getting *big and positive.* In fact,

$$\lim_{x \to 2+} \frac{x^2 + x - 2}{x - 2} = +\infty.$$

Here's why.

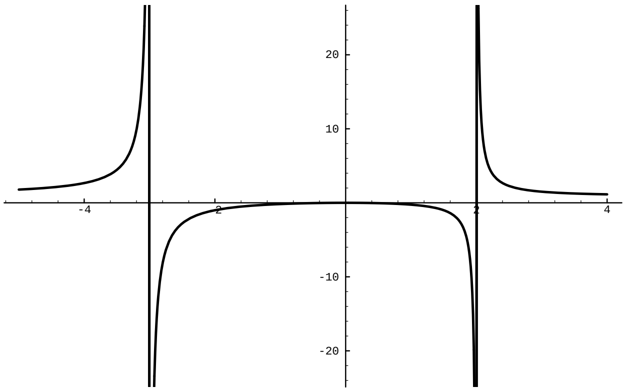
x is approaching 2 *from the right.* Hence, $x - 2$ will be small and positive. On the other hand, $x^2 - 2x - 3 \to 4$. Put these things together. Positive on top plus positive on the bottom means the result is positive. Moreover,

$$\frac{\text{nonzero number}}{\text{little thing}} = \text{big thing}.$$

And $+\infty$ is a " positive big thing". $\square$

Limits at Infinity.

Here is the graph of $f(x) = \dfrac{x^2}{x^2 + x - 6}$:



The graph approaches the horizontal line $y = 1$ as it goes out to the left and right. We write:

$$\lim_{x \to +\infty} \frac{x^2}{x^2 + x - 6} = 1 \text{ and } \lim_{x \to -\infty} \frac{x^2}{x^2 + x - 6} = 1.$$

In general, to say that

$$\lim_{x \to +\infty} f(x) = L$$

means that the graph of $f(x)$ approaches $y = L$ as you plug in larger and larger *positive* values for x.

$$\lim_{x \to -\infty} f(x) = L$$

means that the graph of $f(x)$ approaches $y = L$ as you plug in larger and larger *negative* values for x.

For example, consider $f(x) = \dfrac{x^2}{x^2 + x - 6}$. If we set $x = 10^6$, we get

$$f(x) \approx 0.99999900000699998700005499986700046299873900 4039.$$

That's pretty close to 1, isn't it?

Let's look some examples of a limits at infinity.

Examples.

1.
$$\lim_{x \to +\infty} \frac{7 - x^5}{3x^5 + 4x^3 + 2x^2 + 1}$$

In limits at infinity involving powers of x, the rule of thumb is that the biggest powers dominate. The limit above behaves almost like

$$\lim_{x \to +\infty} \frac{-x^5}{3x^5},$$

because the x^5's on the top and bottom dominate. We expect the answer to be $-\dfrac{1}{3}$.

On way to see this formally is to divide the top and bottom by x^5:

$$\lim_{x \to +\infty} \frac{7 - x^5}{3x^5 + 4x^3 + 2x^2 + 1} = \lim_{x \to +\infty} \frac{\frac{7}{x^5} - 1}{3 + \frac{4}{x^2} + \frac{2}{x^3} + \frac{1}{x^5}}.$$

Now as $x \to +\infty$,

$$\frac{\text{a number}}{x^{positivepower}} \to \frac{\text{a number}}{\text{something humongous}} = 0.$$

Hence,

$$\lim_{x\to+\infty} \frac{\frac{7}{x^5} - 1}{3 + \frac{4}{x^2} + \frac{2}{x^3} + \frac{1}{x^5}} = \lim_{x\to+\infty} \frac{0-1}{3+0+0+0} = -\frac{1}{3}.$$

What else can happen?

$$\lim_{x\to-\infty} \frac{x^{15} - 3x^9 + 47}{x^2 - x + 1} = -\infty,$$

because the x^{15} on top beats out the puny x^2 on the bottom.

By the way, it would be correct to say this limit *diverges*. However, it's more informative to say *how* it diverges. In this case, the function $\dfrac{x^{15} - 3x^9 + 47}{x^2 - x + 1}$ becomes large and negative, so we write $-\infty$ for the limit.

On the other hand,

$$\lim_{x\to+\infty} \frac{2x+4}{x^{5/3} + 7x - 3} = 0,$$

because the $x^{5/3}$ on the bottom beats out the x^1 on the top. □

2.
$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+3}}$$

This one is a little tricky!

Here's the computation. The highest power on the top and the bottom is x (since $\sqrt{x^2}$ looks like x), so divide the top and bottom by x:

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+3}} = \lim_{x\to-\infty} \frac{1}{\frac{1}{x}\sqrt{x^2+3}} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+\frac{3}{x^2}}} = \frac{1}{-1} = -1.$$

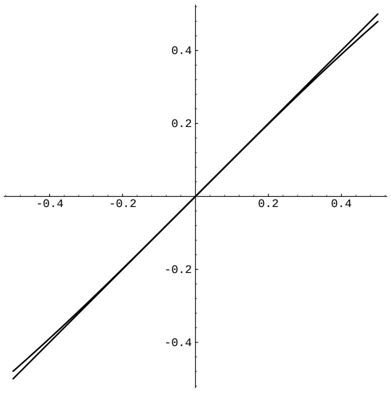
Where did that negative sign come from? Look at the bottom, which was $\dfrac{1}{x}\sqrt{x^2+3}$. x is going to $-\infty$, so x is taking on *negative* values. Now $\sqrt{\cdot}$ is positive, so $\dfrac{1}{x}\sqrt{x^2+3}$ is *negative*.

When you push the $\dfrac{1}{x}$ into the square root, you must leave a negative sign outside. Otherwise, you'd have $\sqrt{\text{junk}}$, a *positive* thing.

This is a case where it matters that x is going to $-\infty$, as opposed to $+\infty$. □

Limits of trig functions.

If you graph $y = \sin x$ and $y = x$, you see that the graphs become almost indistinguishable near $x = 0$.

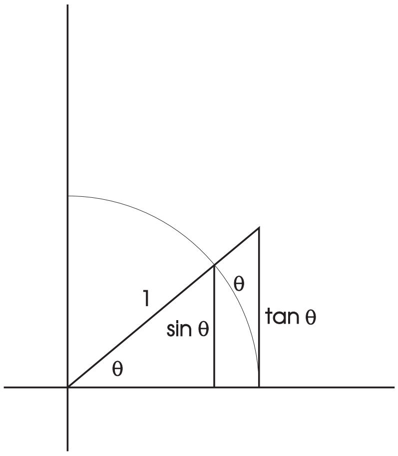


That is, as $x \to 0$, $x \approx \sin x$. This approximation is often used in applications — e.g. analyzing the motion of a simple pendulum for small displacements. We will use it later on to derive the formulas for differentiating trig functions.

We'll usually write the result in this form:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

(Notice that plugging in $x = 0$ gives $\frac{0}{0}$.) We can derive it using the Squeezing Theorem and a little geometry. What we will give is not really a proof from first principles; you can think of it as an argument which makes the result plausible.



We've drawn a sector subtending an angle θ inside a circle of radius 1. (We're using θ instead of x, since θ is more often used for the central angle.) The inner right triangle has altitude $\sin\theta$, while the outer right triangle has altitude $\tan\theta$. The length of an arc of radius 1 and angle θ is just θ.

Clearly,

$$\sin\theta \le \theta \le \tan\theta.$$

Divide through by $\sin\theta$:

$$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}.$$

As $\theta \to 0$, $\dfrac{1}{\cos\theta} \to 1$ — just plug in. By the Squeezing Theorem,

$$\lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1.$$

We get our formula by taking reciprocals.

Examples. The examples below illustrate how our formula can be used. More important than the specific computations is the procedure we're using:

If you want to apply a formula to a situation which doesn't quite fit, try to use algebra to make it fit.

1. $$\lim_{x\to 0} \frac{\sin 4x}{x}.$$

Plugging in $x = 0$ gives $\frac{0}{0}$. We have some work to do.

Our formula looks like this:

$$\lim_{\triangle\to 0} \frac{\sin \triangle}{\triangle} = 1.$$

(I'm using $\triangle$ instead of x to avoid confusing the variable in the formula with the variable in the problem.) *The point is that the thing that is going to 0, the thing inside the sine, and the thing on the bottom must be identical.*

In our problem, there is a $4x$ inside the sine, but an x on the bottom. One or the other must change to match. We don't have nice ways of altering things inside a sine, but making the bottom into $4x$ is easy:

$$\lim_{x\to 0} \frac{\sin 4x}{x} = 4 \lim_{x\to 0} \frac{\sin 4x}{4x}.$$

Let $u = 4x$. As $x \to 0$, $u = 4x \to 0$. So we have

$$4 \lim_{x\to 0} \frac{\sin 4x}{4x} = 4 \lim_{u\to 0} \frac{\sin u}{u} = 4 \cdot 1 = 4. \ \square$$

2. $$\lim_{t\to 0} \frac{\cos 5t - 1}{t}.$$

Plugging in $t = 0$ gives $\frac{0}{0}$. We have some work to do, as usual.

Here the rationale is that our formula has sine but the problem has cosine. How can we get from cosine to sine? The obvious way is to use the identity

$$(\sin\theta)^2 + (\cos\theta)^2 = 1.$$

Rearrange it to say

$$(\cos\theta)^2 - 1 = -(\sin\theta)^2.$$

We can get a $(\cos\theta)^2 - 1$ in our problem by multiplying top and bottom by $\cos 5t + 1$:

$$\lim_{t\to 0} \frac{\cos 5t - 1}{t} = \lim_{t\to 0} \frac{\cos 5t - 1}{t} \cdot \frac{\cos 5t + 1}{\cos 5t + 1} = \lim_{t\to 0} \frac{(\cos 5t)^2 - 1}{t(\cos 5t + 1)} = \lim_{t\to 0} \frac{-(\sin 5t)^2}{t(\cos 5t + 1)}.$$

We force our $\frac{\sin\theta}{\theta}$ form to appear:

$$\lim_{t\to 0} \frac{-(\sin 5t)^2}{t(\cos 5t + 1)} = -\lim_{t\to 0} \frac{\sin 5t}{t} \cdot \frac{\sin 5t}{\cos 5t + 1}.$$

The limit of a product is the product of the limits, so we can do the two terms separately and multiply at the end. First,

$$\lim_{t\to 0} \frac{\sin 5t}{t} = 5 \lim_{t\to 0} \frac{\sin 5t}{5t} = 5 \cdot 1 = 5,$$

using the method of the previous example.

Next,

$$\lim_{t\to 0}\frac{\sin 5t}{\cos 5t+1}=\frac{\sin 0}{\cos 0+1}=0,$$

by plugging in. So

$$-\lim_{t\to 0}\frac{\sin 5t}{t}\cdot\frac{\sin 5t}{\cos 5t+1}=5\cdot 0=0.\ \square$$

3.
$$\lim_{x\to 0}\frac{\sin 7x}{x^2}.$$

Plugging in gives $\frac{0}{0}$. The limit may or may not exist.

We force our $\frac{\sin x}{x}$ form to appear:

$$\lim_{x\to 0}\frac{\sin 7x}{x^2}=7\lim_{x\to 0}\frac{\sin 7x}{7x}\cdot\lim_{x\to 0}\frac{1}{x}.$$

The first limit is 1, but the second is undefined. Therefore, the original limit is undefined. $\square$

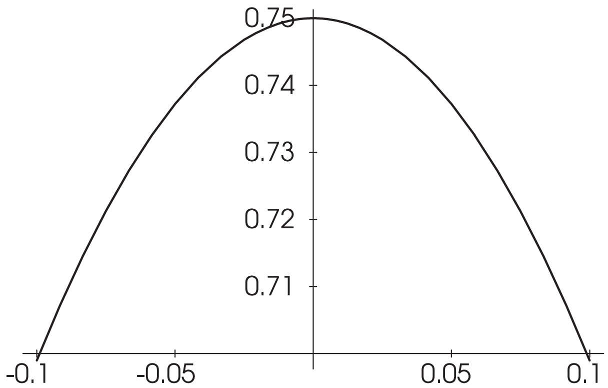
4.
$$\lim_{x\to 0}\frac{\sin 3x}{\tan 4x}.$$

If we set $x=0$, we get $\frac{0}{0}$. Sigh.

Let's see what we can deduce by plotting the graph. The command

```
Plot[Sin[3 x]/Tan[4 x], {x, -0.1, 0.1}]
```

produces the following picture:



Mathematica doesn't even show a "hole" at $x=0$, even though we know the function is undefined there. We suspect that the limit is defined, and the picture suggests that it's around 0.75.

First, we break the tangents down into sines and cosines:

$$\lim_{x\to 0}\frac{\sin 3x}{\tan 4x}=\lim_{x\to 0}(\sin 3x)\frac{\cos 4x}{\sin 4x}.$$

Next, we force our $\frac{\sin\theta}{\theta}$ form to appear. Since we've got $\sin 3x$ and $\sin 4x$, we'll need to make a $3x$ and a $4x$ to match:

$$\lim_{x\to 0}(\sin 3x)\frac{\cos 4x}{\sin 4x}=\frac{3}{4}\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{4x}{\sin 4x}\cdot(\cos 4x).$$

Now we take the limit of each piece:

$$\frac{\sin 3x}{3x} \to 1, \quad \frac{4x}{\sin 4x} \to 1, \quad \cos 4x \to 1.$$

The limit of a product is the product of the limits:

$$\frac{3}{4} \lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{4x}{\sin 4x} \cdot (\cos 4x) = \frac{3}{4} \cdot 1 \cdot 1 \cdot 1 = \frac{3}{4}. \ \square$$

Problems on Limits.

1. Give an example of a function $f(x)$ which is defined for all x and a number c such that

$$\lim_{x\to c} f(x) \neq f(c).$$

2. $\displaystyle\lim_{x\to 2} \frac{x^2-4}{x^2-x-2}$

3. $\displaystyle\lim_{x\to 3} \frac{x^2+2x-3}{x^2-9}$

4. $\displaystyle\lim_{x\to 3-} \frac{x^2+2x-3}{x^2-9}$

5. $\displaystyle\lim_{x\to +\infty} \frac{3x^7-5x^4+x-2}{10-2x^4-3x^7}$

6. $\displaystyle\lim_{x\to +\infty} \frac{3x^7-5x^4+x-2}{10-2x^4-3x^6}$

7. $\displaystyle\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1}$

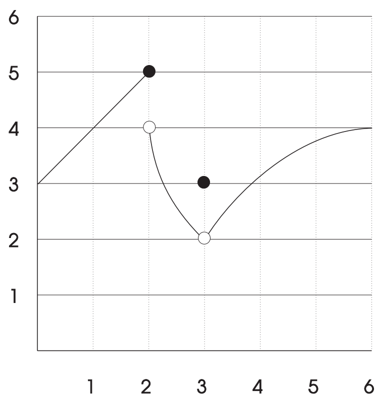
8. $\displaystyle\lim_{x\to 2} \sqrt[5]{\frac{2x+5}{x-7}}$

9. Suppose

$$f(x) = \begin{cases} 6 & \text{if } x \leq 3 \\ \dfrac{x^2-9}{x-3} & \text{if } x > 3 \end{cases}$$

Determine whether $\lim_{x\to 3} f(x)$ is defined. If it is, compute it.

10. The picture below shows that graph of a function $y = f(x)$.



Compute:

(a) $\lim_{x \to 2-} f(x)$ (b) $\lim_{x \to 2+} f(x)$ (c) $\lim_{x \to 2} f(x)$

(d) $\lim_{x \to 3-} f(x)$ (e) $\lim_{x \to 3+} f(x)$ (f) $\lim_{x \to 3} f(x)$

(g) $f(3)$

11. $\lim_{x \to 0} \dfrac{x \sin 3x}{1 - \cos 3x}$

Solutions to the Problems on Limits.

1. Give an example of a function $f(x)$ which is defined for all x and a number c such that

$$\lim_{x \to c} f(x) \neq f(c).$$

There are lots of possible answers to this question. For example, let

$$f(x) = \begin{cases} x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Then

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0, \text{ but } f(0) = 1.$$

Of course, the condition $\lim_{x \to c} f(x) \neq f(c)$ means that the function is **not continuous** at c. The value of $\lim_{x \to c} f(x)$ (or even its existence) does not depend on the value of $f(c)$ (or even its existence). □

2. $$\lim_{x \to 2} \frac{x^2 - 4}{x^2 - x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)(x+1)} = \lim_{x \to 2} \frac{x+2}{x+1} = \frac{4}{3}.$$

Since plugging in $x = 2$ gives $\frac{0}{0}$, it is reasonable to suppose that the zeros are being caused by a *common factor* on the top and the bottom. We factor the top and the bottom and cancel the $x-2$'s. Finally, plugging in $x = 2$ a second time gives $\frac{4}{3}$. Nothing bad happened, so that's the answer. □

3. $$\lim_{x \to 3} \frac{x^2 + 2x - 3}{x^2 - 9} \text{ is undefined.}$$

If we plug in $x = 3$, we get $\frac{12}{0}$, a *nonzero* number over 0. In this case, the limit does not exist.

(By the way, I hope those of you who know L'Hôpital's Rule didn't try to use it here. It won't work.) □

4. $$\lim_{x \to 3-} \frac{x^2 + 2x - 3}{x^2 - 9} = -\infty.$$

This problem is different from the previous one because we're approaching 3 *from the left*. It would not be incorrect to say that the limit is undefined, but it is better to say that the limit is $-\infty$. There is a **vertical asymptote** at $x = 3$, and the graph goes *downward* (to $-\infty$) as it approaches the asymptote from the left.

How do you see algebraically that this is what it does? One way is to plug numbers close to 3, but less than 3, into $\frac{x^2 + 2x - 3}{x^2 - 9}$. For example, if $x = 2.99999$, $\frac{x^2 + 2x - 3}{x^2 - 9} \approx -199999$ — a big negative number. This doesn't *prove* that it's going to $-\infty$, but it's pretty good evidence.

Here is how we can analyze it. We know that plugging in $x = 3$ gives $\frac{12}{0}$. Therefore, we should be getting either $+\infty$ or $-\infty$ — the reciprocal of something small (≈ 0) should be something big.

Now let x approach 3 from the left. $x^2 + 2x - 3$ surely goes to 12, a positive number. $x^2 - 9$ goes to 0, but since $x < 3$, $x^2 - 9$ will be negative. Now $\frac{\text{positive}}{\text{negative}} = \text{negative}$, so our answer is $-\infty$. □

5. $$\lim_{x \to +\infty} \frac{3x^7 - 5x^4 + x - 2}{10 - 2x^4 - 3x^7} = \lim_{x \to +\infty} \frac{3 - \frac{5}{x^3} + \frac{1}{x^6} - \frac{2}{x^7}}{\frac{10}{x^7} - \frac{2}{x^3} - 3} = \frac{3 - 0 + 0 - 0}{0 - 0 - 3} = -1.$$

We divide the top and bottom by the highest power of x that occurs in either — in this case, x^7. Then we use the fact that

$$\frac{\text{number}}{\mathrm{x}^{\text{positivepower}}} \to 0$$

as $x \to \pm\infty$. □

6.
$$\lim_{x\to+\infty} \frac{3x^7 - 5x^4 + x - 2}{10 - 2x^4 - 3x^6} = \lim_{x\to+\infty} \frac{3 - \frac{5}{x^3} + \frac{1}{x^6} - \frac{2}{x^7}}{\frac{10}{x^7} - \frac{2}{x^3} - \frac{3}{x}} = \frac{3 - 0 + 0 - 0}{0 - 0 - 0} = -\infty.$$

There's little question that the limit is undefined; why is it $-\infty$? One approach is to note that the top and bottom are dominated by the highest powers of x. So this limits looks like

$$\lim_{x\to+\infty} \frac{3x^7}{-3x^6} = \lim_{x\to+\infty} -x = -\infty.$$

As a check, if you plug $x = 10^6$ into $\dfrac{3x^7 - 5x^4 + x - 2}{10 - 2x^4 - 3x^6}$, you get (approximately) -10^6. □

7.
$$\lim_{x\to1} \frac{x - 1}{\sqrt{x} - 1} = \lim_{x\to1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{\sqrt{x} - 1} = \lim_{x\to1} (\sqrt{x} + 1) = 1 + 1 = 2.$$

Plugging in $x = 1$ gives $\dfrac{0}{0}$. As usual, we look for a common factor that is producing the zeros. Since the bottom is "smaller" than the top, it's reasonable to see if the bottom is actually a *factor* or the top. It is; we cancel the $\sqrt{x} - 1$'s, and plug in to finish. □

8.
$$\lim_{x\to2} \sqrt[5]{\frac{2x + 5}{x - 7}} = \sqrt[5]{-\frac{9}{5}} \approx -1.12475.$$

Nothing going on here — we just plug in $x = 2$. (By the way — Do you know how to compute $\sqrt[5]{-\dfrac{9}{5}}$ on your calculator?) □

9. Since

$$f(x) = \begin{cases} 6 & \text{if } x \le 3 \\ \dfrac{x^2 - 9}{x - 3} & \text{if } x > 3 \end{cases}$$

is made of "two pieces" glued together at $x = 3$, we compute $\lim_{x\to3} f(x)$ by computing the limits from the left and right, and seeing if they're equal. (If they *aren't* equal, the limit is undefined.)

The left-hand limit is

$$\lim_{x\to3-} f(x) = \lim_{x\to3-} 6 = 6.$$

The right-hand limit is

$$\lim_{x\to3+} f(x) = \lim_{x\to3+} \frac{x^2 - 9}{x - 3} = \lim_{x\to3+} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x\to3+} (x + 3) = 6.$$

Since the left- and right-hand limits are equal, $\lim_{x\to3} f(x)$ is defined, and

$$\lim_{x\to3} f(x) = 6. \ \square$$

10. (a) $\lim_{x\to 2-} f(x) = 5$. □

(b) $\lim_{x\to 2+} f(x) = 4$. □

(c) $\lim_{x\to 2+} f(x)$ is undefined. □

(d) $\lim_{x\to 3-} f(x) = 2$. □

(e) $\lim_{x\to 3+} f(x) = 2$. □

(f) $\lim_{x\to 3} f(x) = 2$. □

(g) $f(3) = 3$. □

11.
$$\lim_{x\to 0} \frac{x\sin 3x}{1-\cos 3x} = \lim_{x\to 0} \frac{x\sin 3x}{1-\cos 3x}\frac{1+\cos 3x}{1+\cos 3x} = \lim_{x\to 0} \frac{(x\sin 3x)(1+\cos 3x)}{1-(\cos 3x)^2} =$$
$$\lim_{x\to 0} \frac{(x\sin 3x)(1+\cos 3x)}{(\sin 3x)^2} = \lim_{x\to 0} \frac{(x)(1+\cos 3x)}{\sin 3x} = \frac{1}{3}\lim_{x\to 0} \frac{3x}{\sin 3x}\cdot(1+\cos 3x) = \frac{1}{3}\cdot 1\cdot 2 = \frac{2}{3}.$$

Plugging in gives $\dfrac{0}{0}$, so we have some work to do.

The idea is to try to get rid of the $1 - \cos 3x$ term which is causing division by 0. To do this, we multiplied top and bottom by $1 + \cos 3x$, then used

$$1 - (\cos\theta)^2 = (\sin\theta)^2$$

to convert cosines to sines.

After cancellation, we rigged up a $3x$ on the top to match the $\sin 3x$ on the bottom. Finally, we applied our $\dfrac{\sin\theta}{\theta}$ rule and plugged in. □