2-12-1994

Problems on Integration.

In Problems 1 - 12, compute the integrals.

1. $\displaystyle\int \sin 3x \cos 5x \, dx.$

2. $\displaystyle\int \tan^{-1}(x+2) \, dx.$

3. $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}} \, dx.$

4. $\displaystyle\int \frac{x^4+3x^2+1}{x(x^2+1)^2} \, dx.$

5. $\displaystyle\int \sin x \; e^{5\cos x} \, dx.$

6. $\displaystyle\int \frac{\cos x}{1+(\sin x)^2} \, dx.$

7. $\displaystyle\int \sqrt{1+\sqrt{x}} \, dx.$

8. $\displaystyle\int (\sin 4x)^3 \, dx.$

9. $\displaystyle\int \frac{e^x}{\sqrt{e^x-3}} \, dx.$

10. $\displaystyle\int \frac{x+4}{x^2-2x} \, dx.$

11. $\displaystyle\int (\sin 5x)^2 \left(1+(\cos 5x)^2\right) \, dx.$

12. $\displaystyle\int_0^2 \frac{x^2}{x^3-1} \, dx.$

13. Does the integral

$$\int_1^\infty \frac{e^x - x}{e^{4x}+x^2} \, dx$$

converge or diverge?

14. Find the volume of the **torus of revolution** generated by revolving the interior of the circle

$$(x-1)^2 + y^2 = 1$$

about the y-axis.

Solutions to the Problems on Integration.

1. $\displaystyle\int \sin 3x \cos 5x \, dx$.

Use integration by parts. Here's the parts table:

$$\begin{array}{ccccc} & \dfrac{d}{dx} & & \int dx \\ + & \sin 3x & & \cos 5x \\ & & \searrow & \\ - & 3\cos 3x & & \dfrac{1}{5}\sin 5x \\ & & \searrow & \\ + & -9\sin 3x & \rightarrow & -\dfrac{1}{25}\cos 5x \end{array}$$

Therefore,

$$\int \sin 3x \cos 5x \, dx = \frac{1}{5}\sin 3x \sin 5x + \frac{3}{25}\cos 3x \cos 5x + \frac{9}{25}\int \sin 3x \cos 5x \, dx,$$

$$\frac{16}{25}\int \sin 3x \cos 5x \, dx = \frac{1}{5}\sin 3x \sin 5x + \frac{3}{25}\cos 3x \cos 5x,$$

$$\int \sin 3x \cos 5x \, dx = \frac{5}{16}\sin 3x \sin 5x + \frac{3}{16}\cos 3x \cos 5x + C. \square$$

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2. $\displaystyle\int \tan^{-1}(x+2) \, dx$.

Use integration by parts. Here's the parts table:

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \int dx \\ + & \tan^{-1}(x+2) & & 1 \\ & & \searrow & \\ - & \dfrac{1}{1+(x+2)^2} & & x \end{array}$$

Therefore,

$$\int \tan^{-1}(x+2) \, dx = x\tan^{-1}(x+2) - \int \frac{x}{1+(x+2)^2} \, dx =$$

$$x\tan^{-1}(x+2) - \int \frac{(x+2)-2}{1+(x+2)^2} \, dx = x\tan^{-1}(x+2) - \int \frac{x+2}{1+(x+2)^2} \, dx + \int \frac{2}{1+(x+2)^2} \, dx.$$

For the first integral, let $u = 1 + (x+2)^2$, so $du = 2(x+2)\,dx$. For the second integral, let $v = x+2$, so $dv = dx$.

$$x\tan^{-1}(x+2) - \frac{1}{2}\int \frac{du}{u} + 2\int \frac{1}{1+v^2} \, dv = x\tan^{-1}(x+2) - \frac{1}{2}\ln|u| + 2\tan^{-1} v + C =$$

$$x\tan^{-1}(x+2) - \frac{1}{2}\ln|1+(x+2)^2| + 2\tan^{-1}(x+2) + C. \square$$

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3. $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx.$

This is a trig substitution. Since I see a $1-x^2$ term, I use a sine substitution, thinking of the identity $1-(\sin\theta)^2 = (\cos\theta)^2$. Let $x = \sin\theta$, so $dx = \cos\theta\,d\theta$:

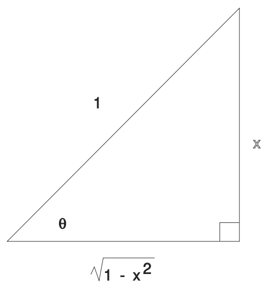
$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{(\sin\theta)^2}{\sqrt{1-(\sin\theta)^2}}\cos\theta\,d\theta =$$

$$\int \frac{(\sin\theta)^2}{\sqrt{(\cos\theta)^2}}\cos\theta\,d\theta = \int \frac{(\sin\theta)^2}{\cos\theta}\cos\theta\,d\theta = \int (\sin\theta)^2\,d\theta.$$

It's an even power of sine, so apply the double angle formula:

$$\frac{1}{2}\int (1-\cos 2\theta)\,d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C = \frac{1}{2}(\theta - \sin\theta\cos\theta) + C.$$

(I used $\sin 2\theta = 2\sin\theta\cos\theta$ to simplify.) Draw the triangle for the substitution:



From the triangle, you can see that

$$\frac{1}{2}(\theta - \sin\theta\cos\theta) + C = \frac{1}{2}\left(\arcsin x - x\sqrt{1-x^2}\right) + C.\ \square$$

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4. $\displaystyle\int \frac{x^4+3x^2+1}{x(x^2+1)^2}\,dx.$

Decompose the integrand using partial fractions:

$$\frac{x^4+3x^2+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2},$$

$$x^4+3x^2+1 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)(x).$$

Setting $x = 0$ yields $1 = A$. Plug this back in and simplify:

$$x^4+3x^2+1 = (x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)(x),$$

$$x^4+3x^2+1 = (x^2+1)^2 + (Bx+C)(x^3+x) + Dx^2 + Ex.$$

Differentiate:

$$4x^3+6x = 4x(x^2+1) + B(x^3+x) + (Bx+C)(3x^2+1) + 2Dx + E.$$

Multiply out the first term on the right and cancel:

$$4x^3 + 6x = 4x^3 + 4x + B(x^3 + x) + (Bx + C)(3x^2 + 1) + 2Dx + E,$$

$$2x = B(x^3 + x) + (Bx + C)(3x^2 + 1) + 2Dx + E. \tag{1}$$

Setting $x = 0$ yields $0 = C + E$. I will call this equation (2).
Differentiate equation (1):

$$2 = B(3x^2 + 1) + B(3x^2 + 1) + (Bx + C)(6x) + 2D,$$

$$2 = 2B(3x^2 + 1) + (Bx + C)(6x) + 2D. \tag{3}$$

Setting $x = 0$ yields

$$2 = 2B + 2D, \quad \text{so} \quad 1 = B + D. \tag{4}$$

Differentiate equation (3):

$$0 = 12Bx + 6(Bx + C) + 6Bx,$$

$$0 = 4Bx + C.$$

Setting $x = 0$ yields $C = 0$, and the equation becomes $0 = 4Bx$. Then setting $x = 1$ gives $B = 0$. By equation (4), $D = 1$. Finally, equation (2) implies that $E = 0$.
This completes the computation of the coefficients.
Plug the numbers back into the decomposition and integrate:

$$\int \frac{x^4 + 3x^2 + 1}{x(x^2 + 1)^2} = \int \left(\frac{1}{x} + \frac{x}{(x^2 + 1)^2}\right) dx.$$

The first term integrates to $\ln|x|$. For the second, let $u = x^2 + 1$, so $du = 2x\,dx$. Then

$$\ln|x| + \frac{1}{2}\int \frac{1}{u^2}\,du = \ln|x| - \frac{1}{2}\frac{1}{u} + C = \ln|x| - \frac{1}{2}\frac{1}{x^2 + 1}. \ \square$$

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5. $\displaystyle\int \sin x\ e^{5\cos x}\,dx.$

Let $u = 5\cos x$ so $du = -5\sin x\,dx$:

$$\int \sin x e^{5\cos x}\,dx = -\frac{1}{5}\int e^u\,du = -\frac{1}{5}e^u + C = -\frac{1}{5}e^{5\cos x} + C. \ \square$$

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6. $\displaystyle\int \frac{\cos x}{1 + (\sin x)^2}\,dx.$

Let $u = \sin x$, $du = \cos x\,dx$:

$$\int \frac{\cos x}{1 + (\sin x)^2}\,dx = \int \frac{1}{1 + u^2}\,du = \tan^{-1} u + C = \tan^{-1}(\sin x) + C. \ \square$$

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7. $\displaystyle\int \sqrt{1 + \sqrt{x}}\,dx.$

This is the exploding power trick! The idea is to force a square into the square root, to kill the square root. Let $x = u^2$, so $dx = 2u\,du$:

$$\int \sqrt{1+\sqrt{x}}\,dx = \int \sqrt{1+u}\cdot(2u)\,du.$$

Now let $w = 1+u$, so $u = w-1$, and $du = dw$:

$$2\int (w-1)\sqrt{w}\,dw = 2\int \left(w^{3/2} - w^{1/2}\right)\,dw = 2\left(\frac{2}{5}w^{5/2} - \frac{2}{3}w^{3/2}\right) + C =$$

$$2\left(\frac{2}{5}(1+u)^{5/2} - \frac{2}{3}(1+u)^{3/2}\right) + C = 2\left(\frac{2}{5}(1+\sqrt{x})^{5/2} - \frac{2}{3}(1+\sqrt{x})^{3/2}\right) + C. \square$$

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8. $\displaystyle\int (\sin 4x)^3\,dx.$

This is an *odd* power of sine. Think of the odd power as having an "extra" sine. Pull it off to make a du, then use $(\sin\theta)^2 + (\cos\theta)^2 = 1$ to turn the remaining sines into cosines:

$$\int (\sin 4x)^3\,dx = \int (\sin 4x)^2 \sin 4x\,dx = \int \left(1 - (\cos 4x)^2\right)\sin 4x\,dx.$$

Now I have a lot of cosines, and a sine to make a du, so let $u = \cos 4x$, giving $du = -4\sin 4x\,dx$:

$$-\frac{1}{4}\int (1-u^2)\,du = -\frac{1}{4}\left(u - \frac{1}{3}u^3\right) + C = -\frac{1}{4}\left(\cos 4x - \frac{1}{3}(\cos 4x)^3\right) + C. \square$$

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9. $\displaystyle\int \frac{e^x}{\sqrt{e^x - 3}}\,dx.$

Let $u = e^x$, so $du = e^x\,dx$:

$$\int \frac{e^x}{\sqrt{e^x-3}}\,dx = \int \frac{u}{\sqrt{u-3}}\,du.$$

Let $w = u-3$, so $u = w+3$, and $dw = du$:

$$\int \frac{w+3}{\sqrt{w}}\,dw = \int \left(\sqrt{w} + \frac{3}{\sqrt{w}}\right)\,dw = \frac{2}{3}w^{3/2} + 6w^{1/2} + C =$$

$$\frac{2}{3}(u-3)^{3/2} + 6(u-3)^{1/2} + C = \frac{2}{3}(e^x-3)^{3/2} + 6(e^x-3)^{1/2} + C. \square$$

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10. $\displaystyle\int \frac{x+4}{x^2-2x}\,dx.$

Decompose the integrand using partial fractions:

$$\frac{x+4}{x^2-2x} = \frac{A}{x} + \frac{B}{x-2},$$

$$x + 4 = A(x-2) + Bx.$$

Setting $x = 0$ yields $A = -2$. Setting $x = 2$ yields $B = 3$. Plug the numbers back in and integrate:

$$\int \frac{x+4}{x^2-2x}\,dx = \int \left(-\frac{2}{x} + \frac{3}{x-2}\right)\,dx = -2\ln|x| + 3\ln|x-2| + C. \square$$

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11. $\displaystyle\int (\sin 5x)^2\left(1 + (\cos 5x)^2\right)\,dx$.

All the powers of sine and cosine are *even*, so apply the double angle formulas for sine and cosine.

$$\int (\sin 5x)^2\left(1 + (\cos 5x)^2\right)\,dx = \int \frac{1}{2}(1 - \cos 10x)\left(1 + \frac{1}{2}(1 + \cos 10x)\right)\,dx =$$

$$\frac{1}{2}\int\left\{1 - \cos 10x + \frac{1}{2}\left(1 - (\cos 10x)^2\right)\right\}\,dx = \frac{1}{2}\int\left\{1 - \cos 10x + \frac{1}{2}(\sin 10x)^2\right\}\,dx =$$

$$\frac{1}{2}\int\left\{1 - \cos 10x + \frac{1}{4}(1 - \cos 20x)\right\}\,dx = \frac{1}{2}\left(x - \frac{1}{10}\sin 10x + \frac{1}{4}x - \frac{1}{80}\sin 20x\right) + C =$$

$$\frac{5}{8}x - \frac{1}{20}\sin 10x - \frac{1}{160}\sin 20x + C. \square$$

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12. $\displaystyle\int_0^2 \frac{x^2}{x^3-1}\,dx$.

I'll compute the antiderivative first. Let $u = x^3 - 1$, so $du = 3x^2\,dx$:

$$\int \frac{x^2}{x^3-1}\,dx = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3-1| + C.$$

Now I'll compute the integral. Note that $\dfrac{x^2}{x^3-1}$ is undefined at $x = 1$, so the integral is improper.

$$\int_0^2 \frac{x^2}{x^3-1}\,dx = \int_0^1 \frac{x^2}{x^3-1}\,dx + \int_1^2 \frac{x^2}{x^3-1}\,dx =$$

$$\lim_{p\to 1-}\int_0^p \frac{x^2}{x^3-1}\,dx + \lim_{q\to 1+}\int_q^2 \frac{x^2}{x^3-1}\,dx =$$

$$\frac{1}{3}\lim_{p\to 1-}\left[\ln|x^3-1|\right]_0^p + \frac{1}{3}\lim_{q\to 1+}\left[\ln|x^3-1|\right]_q^2 =$$

$$\frac{1}{3}\left(\lim_{p\to 1-}\ln|p^3-1| + \lim_{q\to 1+}(\ln 7 - \ln|q^3-1|)\right).$$

Both limits diverge, so the integral diverges. (In fact, if *one* of the limits diverges, the integral is undefined.) Note that you aren't allowed to "cancel" the two limits! $\square$

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13. $\displaystyle\int_1^\infty \frac{e^x - x}{e^{4x} + x^2}\,dx$.

For $x \ge 1$, it follows that

$$\frac{e^x - x}{e^{4x} + x^2} < \frac{e^x - x}{e^{4x}} < \frac{e^x}{e^{4x}} = e^{-3x}.$$

The first inequality holds because removing the x^2 from the bottom of the fraction makes the bottom *smaller*, and that makes the fraction *bigger*. The second inequality holds because I've *added* x to the top (to cancel the $-x$), and doing so makes the fraction bigger.

Now

$$\int_1^\infty e^{-3x}\,dx = \lim_{p\to\infty}\int_1^p e^{-3x}\,dx =$$

$$-\frac{1}{3}\lim_{p\to\infty}\left[e^{-3x}\right]_1^p = -\frac{1}{3}\lim_{p\to\infty}\left(e^{-3p}-e^{-3}\right) = -\frac{1}{3}\cdot\left(-e^{-3}\right) = \frac{e^{-3}}{3}.$$

Since this integral converges, the inequality above shows that $\displaystyle\int_1^\infty \frac{e^x - x}{e^{4x}+x^2}\,dx$ converges as well. □

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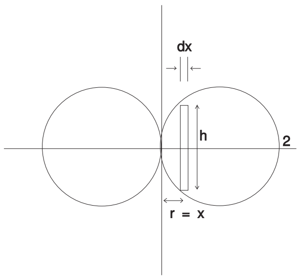
14. Find the volume of the **torus of revolution** generated by revolving the interior of the circle

$$(x-1)^2 + y^2 = 1$$

about the y-axis.



Use cylindrical shells. The radius of a typical shell is x, and the height of a typical shell is $h = 2\sqrt{1-(x-1)^2}$. Therefore, the volume is

$$V = \int_0^2 4\pi x\sqrt{1-(x-1)^2}\,dx.$$

I'll compute the antiderivative separately.

$$\int x\sqrt{1-(x-1)^2}\,dx = \int (x-1+1)\sqrt{1-(x-1)^2}\,dx = \int (x-1)\sqrt{1-(x-1)^2}\,dx + \int \sqrt{1-(x-1)^2}\,dx.$$

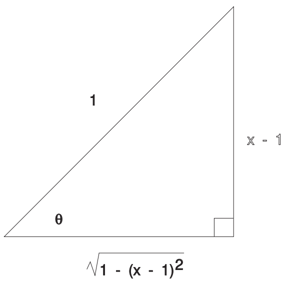
For the first integral, let $u = 1-(x-1)^2$, so $du = -2(x-1)\,dx$. For the second integral, let $x-1 = \sin\theta$, so $dx = \cos\theta\,d\theta$.

$$-\frac{1}{2}\int\sqrt{u}\,du + \int\sqrt{1-(\sin\theta)^2}\cos\theta\,d\theta = -\frac{1}{3}u^{3/2} + \int\sqrt{(\cos\theta)^2}\cos\theta\,d\theta =$$

$$\frac{1}{3}\left(1-(x-1)^2\right)^{3/2} + \int(\cos\theta)^2\,d\theta = -\frac{1}{3}\left(1-(x-1)^2\right)^{3/2} + \frac{1}{2}\int(1+\cos 2\theta)\,d\theta =$$

$$-\frac{1}{3}\left(1-(x-1)^2\right)^{3/2} + \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C = -\frac{1}{3}\left(1-(x-1)^2\right)^{3/2} + \frac{1}{2}\left(\theta + \sin\theta\cos\theta\right) + C.$$

Use the triangle to put the x's back:



$$-\frac{1}{3}\left(1-(x-1)^2\right)^{3/2}+\frac{1}{2}\left(\arcsin(x-1)+\frac{1}{2}(x-1)\sqrt{1-(x-1)^2}\right)+C.$$

Therefore, the volume is

$$V=4\pi\left[-\frac{1}{3}\left(1-(x-1)^2\right)^{3/2}+\frac{1}{2}\left(\arcsin(x-1)+\frac{1}{2}(x-1)\sqrt{1-(x-1)^2}\right)\right]_0^2=2\pi^2\approx 19.73921.\ \square$$