Searching for optimal polygon — remarks about a general construction and application to heptagon and nonagon

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1 Introduction

Results obtained in a first paper, [Dureisseix 97], deal with positioning optimal polygons — i.e. largest regular polygons within a square. Such polygons have two particularities: symmetry with respect to at least one diagonal of the square and the fact that each edge of the square touches one vertex of the polygon.

The technique used for pentagon (building the stellated pentagon) can be generalised for any number of edges. Complete construction in this case still remains open. Nevertheless, it seems necessary to begin with the construction of an angle $\dfrac{\pi}{n}$, which is then enough to build the optimal polygon with the technique proposed herein.

With procedures for folding $\dfrac{\pi}{7}$ and $\dfrac{\pi}{9}$ that are described herein, examples of heptagon and nonagon are performed.

2 General case of an n-edged polygon

In order to get a construction for regular polygon with n edges, the technique used for the pentagon in [Dureisseix 97] can be generalised. Reaching a fraction of the angle $\dfrac{\pi}{n}$ as well as a length are necessary to position the polygon. The idea proposed herein is to take for this length a particular edge of one stellated version of the polygon.

We consider polygons — either stellated or not — for which one edge touches two sides of the square. thanks to previously demonstrated symmetry with respect to one diagonal, at least one edge is parallel to the other diagonal: this is the interesting edge, l_i or l_j, figure 1. One can notice that the largest edge of any stellated versions, l, as been used in [Dureisseix 97] for finding the optimal position of the polygon.

M_i, M_s and M_j denote the vertices of the polygon that touch the sides of the circumscribed square; α_i denotes the angle between $\overrightarrow{M_0M_1}$ and $\overrightarrow{M_0M_i}$, figure 2. It is easy to show that $\alpha_i = \dfrac{\pi}{n}(i-1)$. The searched edge of the stellated polygon has then to satisfy $\alpha_i + \theta_{\text{opt}} = \dfrac{\pi}{4}$ or $\dfrac{3\pi}{4}$. θ_{opt} positions the optimal polygon within the square, for which previous results are in table 1.

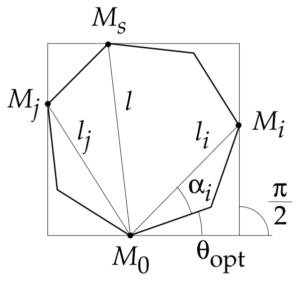

Figure 1: Remarkable edges of stellated polygon

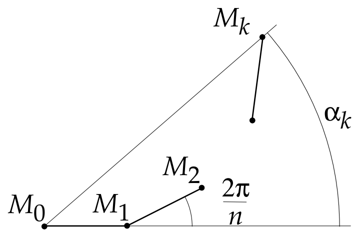

Figure 2: Angles with respect to first edge

2.1 Case of an odd number of edges: $n = 2p + 1$

2.1.1 Particular vertices

Let's find first the vertex M_i, figure 1, that touches the right side of the circumscribed square. i is such that

$$\frac{\pi}{2} \leq \theta_{\text{opt}} + 2(i-1)\frac{\pi}{n} < \frac{\pi}{2} + 2\frac{\pi}{n}$$

and then $i = E(\frac{n}{4} - \theta_{\text{opt}}\frac{n}{2\pi}) + 1$.

Once more, we have to separate sub-cases:

- if p is even: with $\theta_{\text{opt}} = \frac{\pi}{4n}$, $i = E(\frac{p}{2} + \frac{1}{8}) + 1$ and as $p = 2k$,

$$i = k + 1 = \frac{p}{2} + 1 = \frac{n+3}{4}, \quad \alpha_i = \frac{\pi}{4} - \frac{\pi}{4n}$$

- if p is odd, with $\theta_{\text{opt}} = 3\frac{\pi}{4n}$, $i = E(\frac{p}{2} - \frac{1}{8}) + 1$ and as $p = 2k + 1$,

$$i = k + 1 = \frac{p+1}{2} = \frac{n+1}{4}, \quad \alpha_i = \frac{\pi}{4} - 3\frac{\pi}{4n}$$

Each time, we have $\theta_{\text{opt}} + \alpha_i = \frac{\pi}{4}$: this is the searched edge.

It will happen to be interesting to consider also the vertex M_j, figure 1, that touches the left side of the circumscribed square. j is such that

$$3\frac{\pi}{2} \leq \theta_{\text{opt}} + 2(j-1)\frac{\pi}{n} < 3\frac{\pi}{2} + 2\frac{\pi}{n}$$

and then $j = E(\frac{3n}{4} - \theta_{\text{opt}} \frac{n}{2\pi}) + 1$.

- if p is even: with $\theta_{\text{opt}} = \frac{3\pi}{4n}$, $j = E(\frac{3p}{2} + \frac{3}{8}) + 1$ and as $p = 2k$,

$$j = 3k + 1 = \frac{3p}{2} + 1 = \frac{3n+1}{4}, \quad \alpha_j = \frac{3\pi}{4} - \frac{3\pi}{4n}$$

- if p is odd: with $\theta_{\text{opt}} = \frac{\pi}{4n}$, $j = E(\frac{3p}{2} + \frac{5}{8}) + 1$ and as $p = 2k + 1$,

$$j = 3k + 3 = \frac{3(p+1)}{2} = \frac{3(n+1)}{4}, \quad \alpha_j = \frac{3\pi}{4} - \frac{\pi}{4n}$$

We have again $\theta_{\text{opt}} + \alpha_j = \frac{\pi}{4}$: this is an admissible solution again.

Table 1 summarises the situation.

n	p	s	i	α_i	j	α_j	θ_{opt}
$n = 2p$	p pair	$\frac{n}{2}$	$\frac{n}{4} + 1$	$\frac{\pi}{4}$	$\frac{3n}{4} + 1$	$\frac{3\pi}{4}$	0 or $2\frac{\pi}{n}$
	p impair	$\frac{n}{2}$	$\frac{n+2}{4}$	$\frac{\pi}{4} - \frac{\pi}{2n}$	$\frac{3n+2}{4}$	$\frac{3\pi}{4} - \frac{\pi}{2n}$	$\frac{\pi}{2n}$ or $3\frac{\pi}{2n}$
$n = 2p + 1$	p pair	$\frac{n+1}{2}$	$\frac{n+3}{4}$	$\frac{\pi}{4} - \frac{\pi}{4n}$	$\frac{3n+1}{4}$	$\frac{3\pi}{4} - 3\frac{\pi}{4n}$	$\frac{\pi}{4n}$ or $3\frac{\pi}{4n}$
	p impair	$\frac{n+1}{2}$	$\frac{n+1}{4}$	$\frac{\pi}{4} - 3\frac{\pi}{4n}$	$\frac{3(n+1)}{4}$	$\frac{3\pi}{4} - \frac{\pi}{4n}$	$\frac{\pi}{4n}$ or $3\frac{\pi}{4n}$
3	1	2	1		3		$\frac{\pi}{12}$ or $\frac{\pi}{4}$
4	2	2	2		4		0 or $\frac{\pi}{2}$
5	2	3	1		4		$\frac{\pi}{20}$ or $\frac{3\pi}{20}$
6	3	3	2		5		$\frac{\pi}{12}$ or $\frac{\pi}{4}$
7	3	4	2		6		$\frac{\pi}{28}$ or $\frac{3\pi}{28}$
8	4	4	3		7		0 or $\frac{\pi}{4}$
9	4	5	2		7		$\frac{\pi}{36}$ or $\frac{\pi}{12}$

Table 1: Position of particular vertices

2.1.2 Particular edges

Building the interesting edge is easier when the expression of its length is simple. This is the case when one of the vertices M_i or M_j is exactly between vertices M_0 and M_q.

Case where p is even *Case where p is odd*

Figure 3: Remarkable edge when n is odd

- if p is even: the best vertex is M_j, figure 3.

  $$j - s = \frac{n-1}{4} = n - j \quad \text{and then} \quad \beta_j = \alpha_j - \alpha_s = \frac{\pi}{n}(j-s) = \frac{\pi}{4} - \frac{\pi}{4n}$$

  and

  $$l_j = \frac{l}{2\cos\beta_j} = \frac{A/2}{\cos\frac{\pi}{4n}\cos(\frac{\pi}{4} - \frac{\pi}{4n})}$$

- if p is odd: the best vertex is M_i, figure 3.

  $$i - 0 = \frac{n+1}{4} = s - i \quad \text{and then} \quad \beta_i = \alpha_s - \alpha_i = \frac{\pi}{n}(s-i) = \frac{\pi}{4} + \frac{\pi}{4n}$$

  and

  $$l_i = \frac{l}{2\cos\beta_i} = \frac{A/2}{\cos\frac{\pi}{4n}\cos(\frac{\pi}{4} + \frac{\pi}{4n})}$$

For each case, folding the interesting length depends now of the initial construction of angle $\frac{\pi}{4n}$, figure 4. If p is even, the construction follows the pattern of the figure 5, else, it follows the pattern of the figure 6.

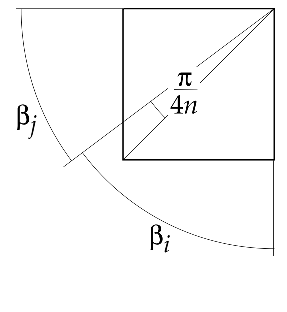

Figure 4: Initial angle $\frac{\pi}{4n}$

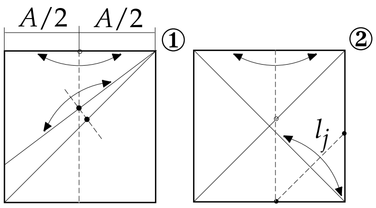

Figure 5: Construction when n is odd and p is even

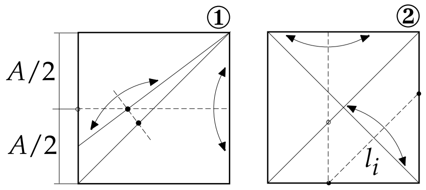

Figure 6: Construction when n and p are odd

This construction can be realised more or less easily for several particular cases.

2.2 Case of an even number of edges: $n = 2p$

Case for which n is even has similar framework of folding. Now, $i = E(\frac{p}{2}) + 1$, $j = E(\frac{3p}{2}) + 1$.

2.2.1 Particular vertices

- If p is even: with $\theta_{\text{opt}} = 0$ and $p = 2k$,

$$\begin{aligned} i &= k+1 = \frac{p}{2}+1 = \frac{n}{4}+1, & \alpha_i &= \frac{\pi}{4} \\ j &= 3k+1 = \frac{3p}{2}+1 = \frac{3n}{4}+1, & \alpha_j &= \frac{3\pi}{4} \end{aligned}$$

- If p is odd: with $\theta_{\text{opt}} = \dfrac{\pi}{2n}$ and $p = 2k+1$,

$$\begin{aligned} i &= k+1 = \frac{p+1}{2} = \frac{n+2}{4}, & \alpha_i &= \frac{\pi}{4} - \frac{\pi}{2n} \\ j &= 3k+2 = \frac{3p+1}{2} = \frac{3n+2}{4}, & \alpha_j &= \frac{3\pi}{4} - \frac{\pi}{2n} \end{aligned}$$

These results are also collected in table 1.

2.2.2 Particular edges

Now, interesting lengths correspond to vertex M_i. They are more complex than those used when n was odd.

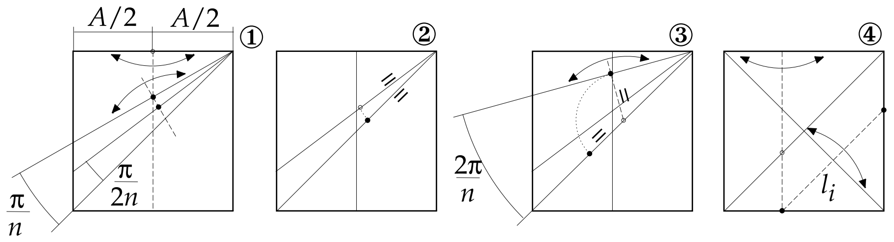

Figure 7: Construction when n and p are even

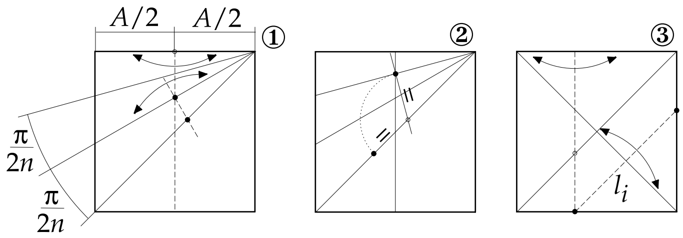

Figure 8: Construction when n is even and p is odd

- If p is even, $i - 0 = \frac{n}{4} + 1$ and $s - i = \frac{n}{4} - 1$. With figure 9, we can find

  $$\beta_i = \alpha_s - \alpha_i = \frac{\pi}{n}(s-i) = \frac{\pi}{4} - \frac{\pi}{n}$$

  and

  $$l_i = \frac{A/2}{\cos\frac{\pi}{2n}\cos(\frac{\pi}{4} - \frac{\pi}{n})}(1 + \sin 2\frac{\pi}{n})$$

  Figure 7 shows the construction of this edge.

- If p is odd, $i - 0 = \frac{n}{4} + \frac{1}{2}$ and $s - i = \frac{n}{4} - \frac{1}{2}$. With figure 9, we can find

  $$\beta_i = \alpha_s - \alpha_i = \frac{\pi}{n}(s-i) = \frac{\pi}{4} - \frac{\pi}{2n}$$

  and

  $$l_i = \frac{A/2}{\cos\frac{\pi}{2n}\cos(\frac{\pi}{4} - \frac{\pi}{2n})}(1 + \sin 2\frac{\pi}{n})$$

  Figure 8 shows the construction of this edge.

3 Example of the heptagon

With the previous notations, $n = 7 = 2p + 1$ with $p = 3 = 2k + 1$ and $k = 1$. Interesting vertex is M_i with $i = 2$; it is built from the length l_i.

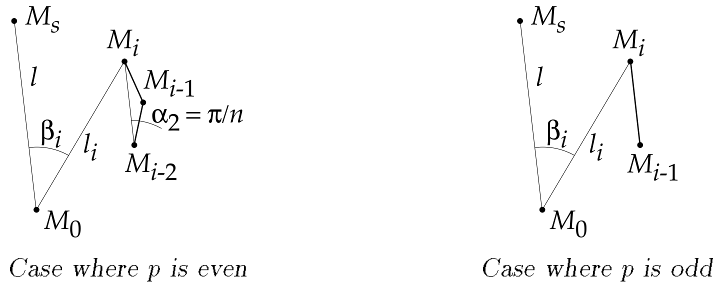

Figure 9: Remarkable edge when n is even

Here, the angle $\frac{\pi}{4n} = \frac{\pi}{28}$ is folded with the technique used in [Geretschläger 97]. One can notice that this construction is non euclidean: it is impossible to perform it with the only tools that are compass and straight edge. Figure 10 shows the whole construction.

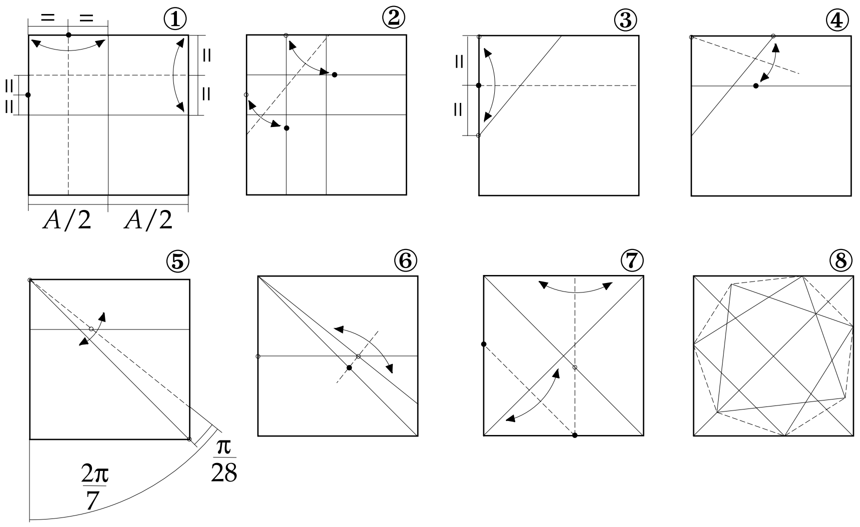

Figure 10: Finding the optimal heptagon

Stages 1 and 2: building a fold with slope $2\cos\frac{2\pi}{7}$.

Stages 3 to 5: building the angle $\frac{\pi}{28}$.

Stages 6 to 8: construction the heptagon.

4 Example of the nonagon

This time, $n = 9 = 2p + 1$ with $p = 4 = 2k$ and $k = 2$. Interesting vertex is again M_i with $i = 2$.

The angle $\frac{\pi}{4n} = \frac{\pi}{36}$ is folded with the trisection classical technique (see for instance [Fusimi 80], [Hull 96]); it is again a non euclidean technique. Figure 11 shows the whole construction.

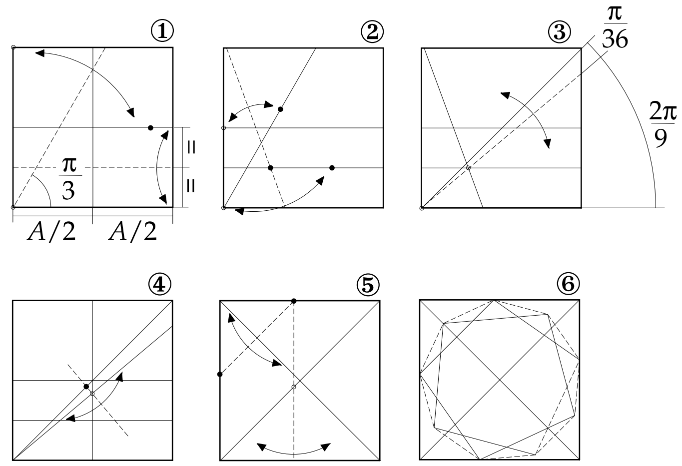

Figure 11: Finding the optimal nonagon

Stage 1: building the angle $\frac{\pi}{3}$.

Stages 2 to 3: trisection of the angle, to obtain $\frac{\pi}{36}$.

Stages 4 to 6: building the nonagon.

4.1 Example of the hexagon

Cases where n is even are simpler to perform, at least for small values of n. The previous technique is still applicable, and the construction can often be simplified by use of special properties of corresponding angles.

The hexagon case is presented as an example in the figure 12.

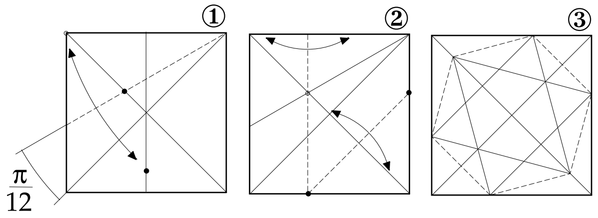

Figure 12: Finding the optimal hexagon

5 Conclusions

Optimal polygon constructions are precise and simple when using stellated particular versions.

General case of a polygon with n edges is performed as soon as the angle $\frac{\pi}{n}$ have been built. With small values of n, it is easy to obtain; this allowed the folding of polygons till the nonagon.

References

[Dureisseix 97] Dureisseix David, à la recherche du polygone optimal — application au pentagone, september 1997, unpublished note.

[Fusimi 80] Fusimi K., Trisection of an angle by H. Abe, *Science of Origami*, october 1980, p. 8, supplement to Saiensu.

[Geretschläger 97] Geretschläger Robert, Folding the regular heptagon, *Crux Mathematicorum*, vol. 23, number 2, march 1997, pp. 81–88.

[Hull 96] Hull Thomas, A note on impossible paperfolding, *American Mathematical Monthly*, vol. 103, number 3, march 1996, pp. 242–243.