# Searching for optimal polygon — application to the pentagon case

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## 1 Introduction

Besides classical beginning square, several models are built from various different forms of paper. Among them, regular polygons are often used for geometric models or modules. A usual feature related to this kind of constructions is to get the largest regular polygon within a given square. A large number of folding procedures, exact or approximated, simple or complex, using more or less of the paper area, exists.

The goal is herein to find the optimal polygon (i.e. the regular polygon which size is the largest one, within a given square). Several folding procedures are compared for the pentagon case; moreover, a new exact and optimal construction is proposed, while trigonometry is the main tool needed as well as the integer part.

## 2 Construction principle

One can easily show that at least one vertex of optimal polygon intersect (in a large sense) an edge of the square. An auxiliary question is the following: find the smallest rectangle circumscribed to the given polygon. Thought this question is more general, the answer is paradoxically easier to find than these related to the circumscribed square. This last case is a particular one; the previous answer will then

$H$ and $B$ will denote the height and width of circumscribed rectangles, $n$, the number of edges for the regular polygon and $R$, the radius of the circumscribed circle. The $\theta$-angle positions the polygon from the considered circumscribed rectangle, as described by the figure 1.

This problem posses similarities with the problem of flow irregularities of axial piston hydrostatic components for power fluid transmissions [Fayet 91], [Cognard 93]. Cases where $n$ is odd or even have to be separated, as we do herein.

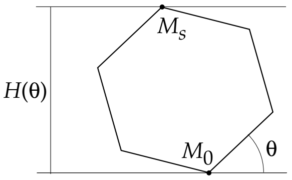



Figure 1: Construction principle

## 2.1 Case of an odd number of edges

Figure 2 indicates notations used. It is then obvious that minimum height for the circumscribed rectangle to the $\theta$-inclined polygon is:

$$\text{For} \quad \theta \in [\frac{\pi}{n}, 2\frac{\pi}{n}], \quad H(\theta) = l\cos(\theta - 3\frac{\pi}{2n})$$

Moreover, this function is $\frac{\pi}{n}$-periodic as shown in the figure 3. It is then easy to show that the number of the vertex $M_s$ for the rectangle height is $s = \frac{n+1}{2}$.

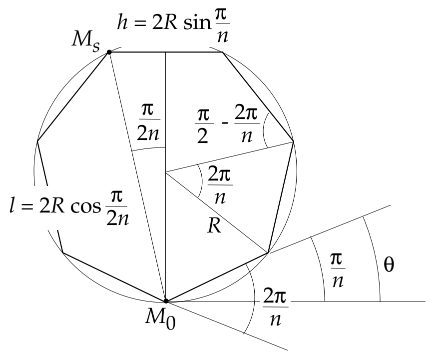



Figure 2: Odd case notations

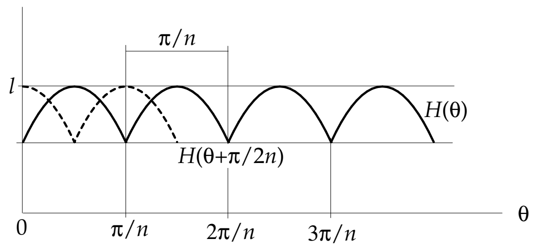



Figure 3: Characteristic function for odd case

The corresponding width is get as well:

$$B(\theta) = H(\theta - \frac{\pi}{2})$$

In order to get a phase shift within $[0, \frac{\pi}{n}]$, one has to use the integer part function $E$:

$$q = E(\frac{n}{2} + 1)$$

such as $\frac{n}{2} < q \leq \frac{n}{2} + 1$, and $0 < \phi = \frac{\pi}{n}q - \frac{\pi}{2} \leq \frac{\pi}{n}$, and then $B(\theta) = H(\theta + \phi)$.

Herein, $n$ is odd, so: $q = \frac{n}{2} + \frac{1}{2}, \quad \phi = \frac{\pi}{2n}$.

For a given position $\theta$, $H(\theta)$ and $B(\theta)$ are the minima height and width respectively. Using a circumscribed rectangle leads to two uncoupled minimisation problems. For the special case of the square, itself has to be circumscribed to the rectangle; its edge is then $A(\theta) = \max(H(\theta), B(\theta))$. The smallest square is get for the position $\theta_{\text{opt}}$:

$$A_{\text{opt}} = \min_{\theta} A(\theta) = A(\theta_{\text{opt}})$$

Here, the figure 3 illustrates the result:

$$\theta_{\text{opt}} = \frac{\pi}{4n} \quad \text{or} \quad 3\frac{\pi}{4n}, \quad \text{and} \quad A_{\text{opt}} = l \cos \frac{\pi}{4n}$$

## 2.2 Case of an even number of edges

This time, $n = 2p$, where $p$ is an integer. The configuration is given on the figure 4.

$$\text{For} \quad \theta \in [0, 2\frac{\pi}{n}], \quad H(\theta) = l' \cos(\theta - \frac{\pi}{n})$$

the function is moreover $2\frac{\pi}{n}$-periodic, figure 5. This time, the number of vertex $M_s$ used for the rectangle height is $s = \frac{n}{2}$; it is opposed to vertex $M_0$.

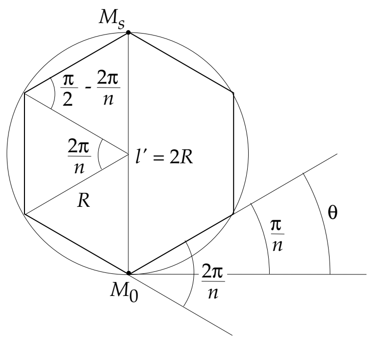



Figure 4: Even case notations

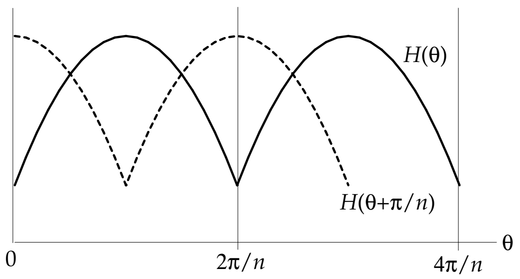



Figure 5: Characteristic function for even case

As

$$B(\theta) = H(\theta - \frac{\pi}{2}) = H(\theta + \phi')$$

the result is this time $\phi' = 2\frac{\pi}{n}q' - \frac{\pi}{2} \quad \in \quad ]0, 2\frac{\pi}{n}]$ and $q' = E(\frac{n}{4}+1)$.

2 sub-cases have to be treated:

- if $p$ is even: $p = 2k$, $n = 4k$ and

$$q' = E(k+1) = k+1 = \frac{n}{4}, \quad \phi' = \frac{2\pi}{4k} - \frac{\pi}{2} = \frac{\pi}{2k} = 2\frac{\pi}{n}, \quad B(\theta) = H(\theta)$$

  Then (figure 5)

$$\theta_{\text{opt}} = 0 \quad \text{or} \quad 2\frac{\pi}{n}, \quad \text{and} \quad A_{\text{opt}} = H(\theta_{\text{opt}}) = l' \cos\frac{\pi}{n}$$

- if $p$ is odd : $p = 2k+1$, $n = 4k+2$ et

$$q' = E(k+\frac{3}{2}) = k+1 = \frac{n+2}{4}, \quad \phi' = \frac{\pi}{2(2k+1)} = \frac{\pi}{n}, \quad B(\theta) = H(\theta + \frac{\pi}{n})$$

  Then (figure 5)

$$\theta_{\text{opt}} = \frac{\pi}{n} \quad \text{or} \quad 3\frac{\pi}{n}, \quad \text{and} \quad A_{\text{opt}} = H(\theta_{\text{opt}}) = l' \cos\frac{\pi}{2n}$$

**Remark:** as for each case, the minimum is obtained for $H(\theta) = B(\theta)$, each edge of the circumscribed square touches a vertex of the optimal polygon.

# 3 Results

In order to quantify the fact that the polygon is or is not the largest one, an indicator, $\lambda$, is built as the ratio between areas:

$$\lambda = \frac{S_p}{S_c} \leq 1$$

where $S_c$ is the square area and $S_p$, the area of the polygon.

$$S_c = A_{\text{opt}}^2, \quad S_p = nR^2 \sin\frac{\pi}{n} \cos\frac{\pi}{n}$$

Table 1 summarises the previous results. It is easy to show that each polygon has one diagonal as symmetry axis.

# 4 Example of the pentagon

The pentagon is a classical example, it has been several times treated. Remarkable quantities are:

$$\sin\frac{\pi}{5} = \frac{\sqrt{10-2\sqrt{5}}}{4} \qquad \cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4} = \frac{1}{\sqrt{5}-1}$$

$$\tan\frac{\pi}{5} = \sqrt{5-2\sqrt{5}} \qquad \sin\frac{2\pi}{5} = \frac{\sqrt{10+2\sqrt{5}}}{8}$$

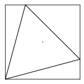
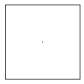
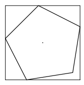
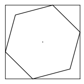
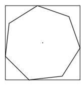
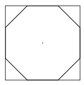
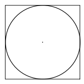
| $n$ | $A_{\text{opt}}/2R$ | $\theta_{\text{opt}}$ | $\lambda_{\text{opt}}$ | |
|---|---|---|---|---|
| odd | $\cos\frac{\pi}{2n}\cos\frac{\pi}{4n}$ | $\frac{\pi}{4n}$ or $\frac{3\pi}{4n}$ | $\frac{n}{4}\frac{\sin\frac{\pi}{n}\cos\frac{\pi}{n}}{\cos^2\frac{\pi}{2n}\cos^2\frac{\pi}{4n}}$ | |
| even, multiple of 4 | $\cos\frac{\pi}{n}$ | $0$ or $\frac{2\pi}{n}$ | $\frac{n}{4}\tan\frac{\pi}{n}$ | |
| even, non multiple of 4 | $\cos\frac{\pi}{2n}$ | $\frac{\pi}{2n}$ or $\frac{3\pi}{2n}$ | $\frac{n}{4}\frac{\sin\frac{\pi}{n}\cos\frac{\pi}{n}}{\cos^2\frac{\pi}{2n}}$ | |
| 3 | 0.83652 | $\frac{\pi}{12}$ or $\frac{\pi}{4}$ | 0.46410 |  |
| 4 | $\frac{\sqrt{2}}{2} \approx 0.70711$ | $0$ or $\frac{\pi}{2}$ | 1 |  |
| 5 | 0.93935 | $\frac{\pi}{20}$ or $\frac{3\pi}{20}$ | 0.67365 |  |
| 6 | 0.96593 | $\frac{\pi}{12}$ or $\frac{\pi}{4}$ | 0.69615 |  |
| 7 | 0.96880 | $\frac{\pi}{28}$ or $\frac{3\pi}{28}$ | 0.72888 |  |
| 8 | 0.92388 | $0$ or $\frac{\pi}{4}$ | 0.82843 |  |
| $\infty$ | 1 | 0 | $\frac{\pi}{4} \approx 0.78539$ |  |

Table 1: Results

as well as the golden ratio

$$h/l = \frac{\sqrt{5}-1}{2}$$

used for exact construction techniques.

Following references are obviously not exhaustive; they refer to one existing presentation of the quoted technique. Some of these techniques are presented herein for the pentagon. They have been designed also in more general cases, for instance in [Hilton and Pederson 83], [Gibbs 83], [Scimemi 89], [Huzita 94], [Geretschläger 97a], [Geretschläger 97b].

## 4.1 Exact constructions

- Quoted from [Kasahara 89], page 253, figure 6.

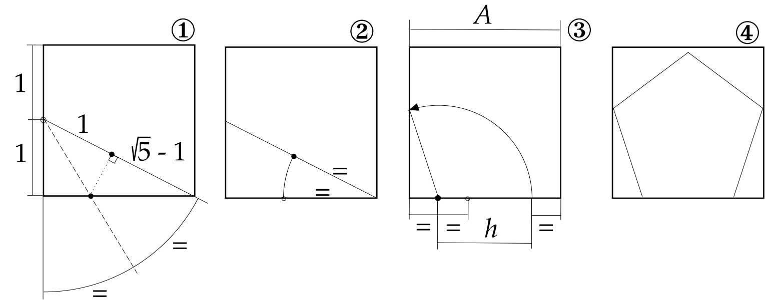



Figure 6:

- Quoted from [Kasahara 89], page 89, figure 7.

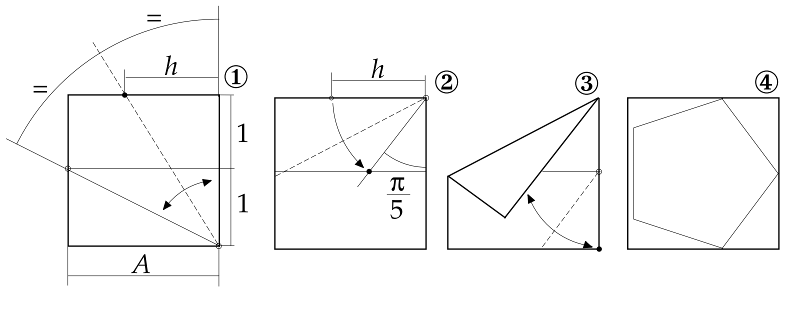



Figure 7:

There two constructions are characterised by $A = 2$, $h = \sqrt{5} - 1 \approx 1.23607$, $\lambda = \frac{5}{4}\frac{h^2}{A^2 \tan\frac{\pi}{5}} \approx 0.65716$. They are not optimal ones, as $\lambda < \lambda_{\text{opt}}$.

## 4.2 Approximated constructions

- "American" technique, [Kasahara 88], page 77, [Kasahara 89], page 252, figure 8. $A = 2$, $R = \dfrac{5\sqrt{5}}{12} \approx 0.93169$, $\lambda = \dfrac{5(4+5\sqrt{10})}{192} \approx 0.51592$.

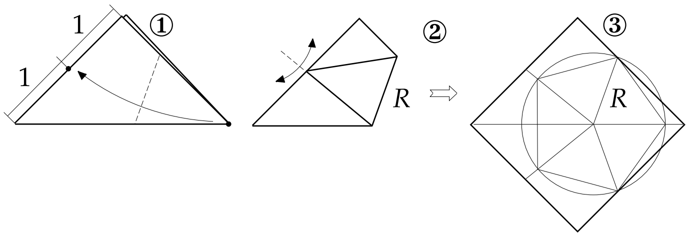



Figure 8:

- T. Kawai method, [Kawai 70] page 124, [Morassi 89], figure 9. $A = 2$, $R = 1$, $\lambda = \dfrac{3(4+5\sqrt{10})}{100} \approx 0.594342$.

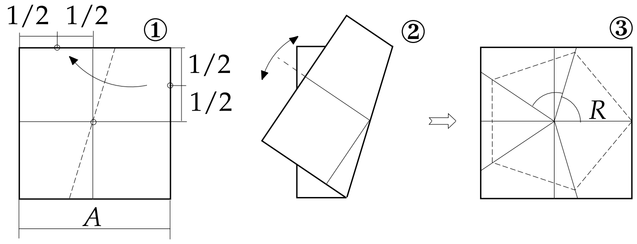



Figure 9:

- Quoted from [Kasahara and Takahama 88], page 64, figure 10. $A = 2$, $h = \dfrac{5}{4} = 1.25$, $\lambda = \dfrac{3}{16} + \dfrac{13\sqrt{91}}{256} \approx 0.67192$.

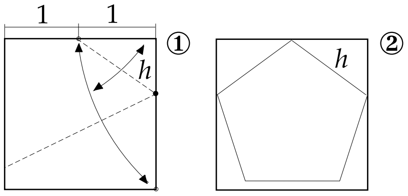



Figure 10:

For which an other construction can be found in [Kasahara 88], page 73, figure 11.

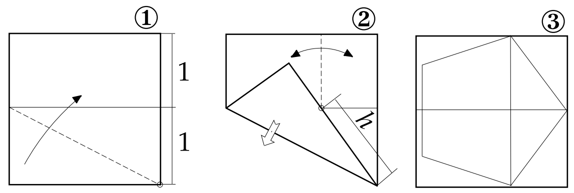



Figure 11:

- Quoted from [Kasahara 89], page 254, figure 12. $A = 2$, $h = \sqrt{\dfrac{5-\sqrt{5}}{2}} \approx 1.17557$, $\lambda \approx 0.59152$.

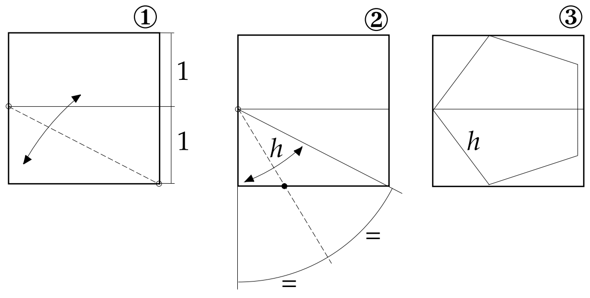



Figure 12:

- Kazuo Haga approach, [Kasahara 88], page 76, figure 13. $A = 2$, $h = \dfrac{5}{4} = 1.25$, $\lambda \approx 0.67203$.

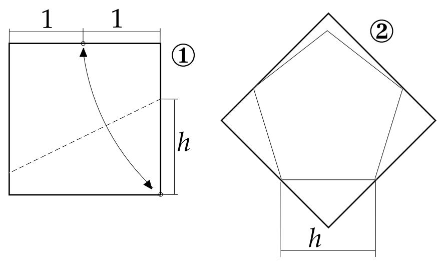



Figure 13:

- Traditional Japanese method, [Kasahara 88], page 75, [Yoshizawa 11] page 26, figure 14. $A = 2$, $R = \dfrac{5\sqrt{2}}{7} \approx 1.01015$, $\lambda = \dfrac{3(4+5\sqrt{10})}{98} \approx 0.60647$.

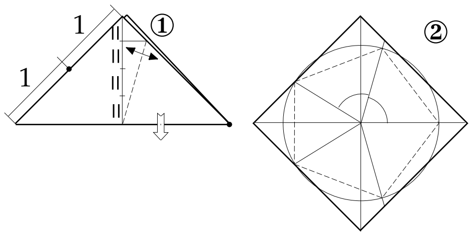



Figure 14:

- Improved traditional Japanese method, [Kasahara 88], page 75, figure 15. $A = 2$, $R = \sqrt{2(10 - 3\sqrt{10})} \approx 1.01308$, $\lambda \approx 0.61006$.

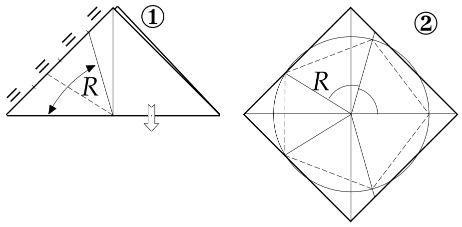



Figure 15:

- R. Morassi method, [Morassi 89], figure 16. $A = 2$, $R = \sqrt{\dfrac{10 - 4\sqrt{3}}{3}} \approx 1.01189$, $\lambda \approx 0.52822$.

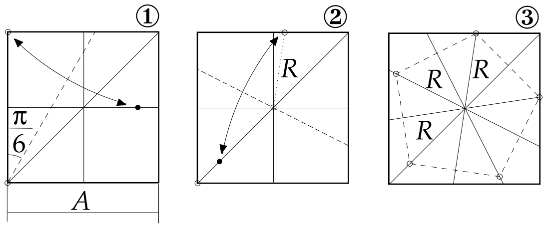



Figure 16:

- F. Rohm method, [Morassi 89], figure 17. $A = 2$, $\lambda \approx 0.808152$.

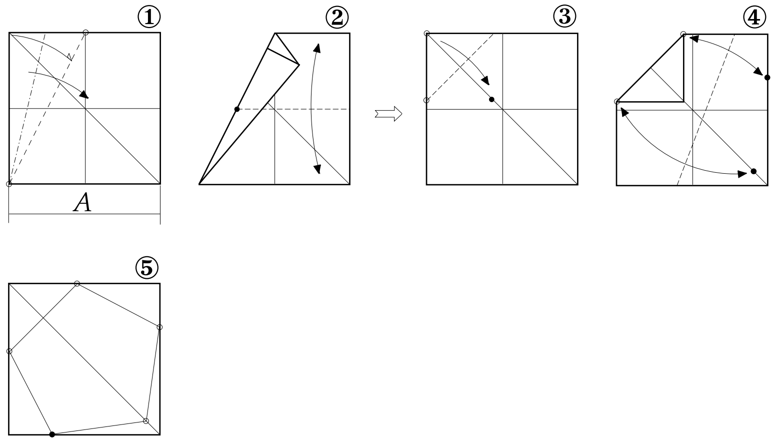



Figure 17:

### 4.3 Exact constructions for optimal pentagon

Historically, the first exact construction for the pentagon has to be credited to [Morassi 89], figure 18. The technique proposed herein has been developed inde-

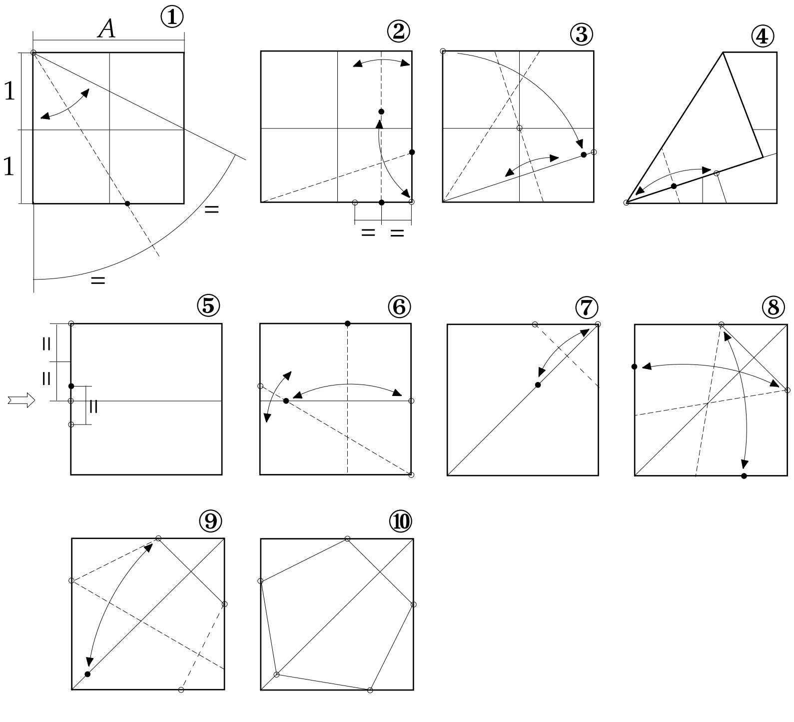



Figure 18:

pendently from the previous one. She leads to a little simpler construction, figure 19, while using the property: $l\cos\dfrac{\pi}{20} = A$.
Steps 1 and 2: folding the golden ratio.
Step 3: building the angles $\pi/5$, $\pi/10$, $\pi/20$.

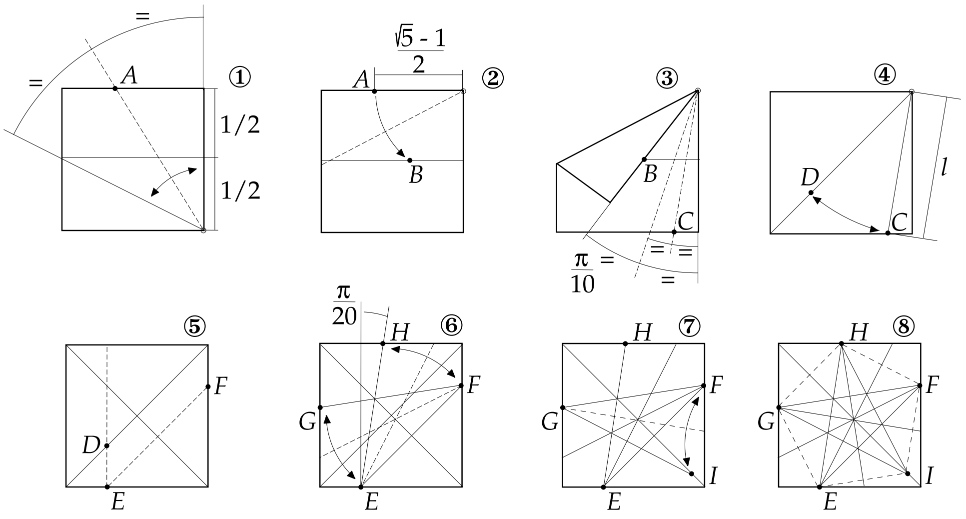



Figure 19:

Step 4: building $l = A/(\cos(\pi/20))$.
Step 5: one edge of the stellated pentagon ...
Steps 6 to 8: ... and the others.

These folding constructions are effectively optimal, as the indicator is $\lambda = \lambda_{\text{opt}} \approx 0.67365$.

# References


[Cognard 93] Cognard Jean-Yves, *Transmission et transformation de puissance*, ENS de Cachan, Département de Génie Mécanique, 1993.

[Fayet 91] Fayet Georges, *Hydraulique - machines et composants*, Paris, Eyrolles, 1991.

[Geretschläger 97a] Geretschläger Robert, Folding the regular heptagon, *Crux Mathematicorum*, vol. 23, number 2, march 1997, pp. 81–88.

[Geretschläger 97b] Geretschläger Robert, Folding the regular nonagon, *Crux Mathematicorum*, vol. 23, number 4, mai 1997, pp. 210–217.

[Gibbs 83] Gibbs W., Paper polygons, *Mathematics Teaching*, vol. 103, juin 1983, pp. 16–17.

[Hilton and Pederson 83] Hilton Peter and Pederson Jean, Approximating any regular polygon by folding paper, *Mathematics Magazine*, vol. 56, number 3, mai 1983, pp. 141–155.

[Huzita 89] Huzita H. (edited by), *Proceedings of the First International Meeting of Origami Science and Technology*, Ferrara, 1989.

[Huzita 94] Huzita Humiaki, Drawing the regular heptagon and the regular nonagon by origami (paper folding), *Symmetry: Culture and Science*, vol. 5, number 1, 1994, pp. 69–84.

[Kasahara 88] Kasahara Kunihiko, *Origami Omnibus: Paperfolding for Everybody*, Tokyo-New York, Japan Publications Inc., 1988.

[Kasahara 89] Kasahara Kunihiko, *El Mundo Nuevo*, Sanrio, 1989.

[Kasahara and Takahama 88] Kasahara Kunihiko and Takahama Toshie, *Origami for the Connoisseur*, Tokyo-New York, Japan Publications Inc., 1987-1988.

[Kawai 70] Kawai Toyoaki, *Origami*, 1970.

[Morassi 89] Morassi Roberto, The elusive pentagon, In Huzita [Huzita 89], pp. 27–37.

[Scimemi 89] Scimemi Benedetto, Draw of a regular heptagon by the folding, In Huzita [Huzita 89], pp. 71–77.

[Yoshizawa 11] Yoshizawa Akira, *Origami Dokuhon*, 1911.