On the Mathematics of Flat Origamis

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Abstract

Origami is the art of folding pieces of paper into works of sculpture without the aid of scissors or glue. Modern advancements in the complexity of origami (e.g., the work of Montroll and Maekawa) reveal a rich geometric structure governing the possibilities of paperfolding. In this paper we initiate a mathematical study of this "origami geometry" and explore the possibilities of a graph theoretic model. In particular, we study the properties of origami models which fold flat (i.e., can be pressed in a book without crumpling). Necessary and sufficient conditions are given for an origami model to locally fold flat, and the problems encountered in trying to extend these results globally are discussed.

1 Introduction

Origami is the art of folding paper, without the aid of scissors or glue, into pieces of sculpture. Even when we restrict ourselves to folding a single square of paper, the possible results offer rich variety and complexity (see [3], [8], [10], or [11]). Things explode further when more than one sheet of paper is employed (see [2]).

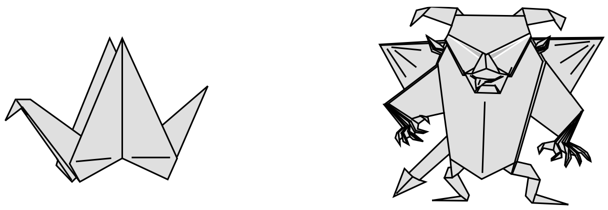

Figure 1.1: two origamis folded from single, uncut squares,
the traditional crane and Jun Maekawa's winged demon.

People, both non-mathematicians and non-origamists, intuitively associate origami with geometry. Despite this, very little on the mathematics of origami has been published. Advanced origamists have provided some results ([3]), particularly in developing techniques to generate ever-more complex and realistic models ([1], [8]). But very few mathematicians, with the notable exceptions of Maekawa and Kawasaki, have investigated a straightforward mathematical analysis of origami. This paper presents such an analysis, and at the same time proves some results commonly known to origamists. A slightly different approach can be found in [5] and [7].

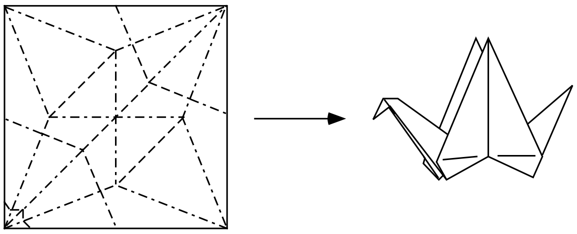

Figure 1.2: the crease-pattern for the traditional crane

When investigating origami, the *crease-pattern* of a model will be our main tool. Figure 1.2 shows the traditional Japanese crane together with its crease-pattern, which can serve as instructions for making the model. These crease-patterns use two different types of dashed lines to represent the two types of creases: *mountain creases* are drawn with a dot-dash-dot line, and *valley creases* are drawn with a dashed line (see Figure 1.3).

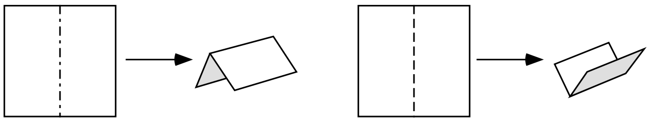

Figure 1.3: a mountain crease (left) and a valley crease (right).

2 Defining Origami

Since anyone familiar with paperfolding has a clear sense of what origami should be, we create a definition which attempts to model the physical situation of folding paper.

We shall define an origami by a pair $(\mathcal{C}, f)$ where $\mathcal{C}$ is a set of lines in the unit square $[0,1] \times [0,1]$, which define the creases, and f is a function $f : \mathcal{C} \to (-\pi, \pi)$ which indicates how far, and in what direction, each crease is folded. This pair $(\mathcal{C}, f)$ thus defines a function from the square $[0,1] \times [0,1]$ into $\mathbf{R}^3$, and in order to guarantee that this function does not necessitate ripping the paper we must impose that this function is 1-1. We formalize this in a definition (following [5]).

Definition: An *origami* is a pair $(\mathcal{C}, f)$, where $\mathcal{C}$ is the crease set and $f : \mathcal{C} \to (-\pi, \pi)$, such that the mapping $(\mathcal{C}, f)$ induces from $[0,1] \times [0,1]$ into $\mathbf{R}^3$ is 1-1.

Note that for a crease line l_1, $f(l_1) > 0$ means that l_1 is a valley and $f(l_1) < 0$ means that l_1 is a mountain. Figure 2.1 shows an origami which folds the square into a 3-D "corner." (Taken from [4].)

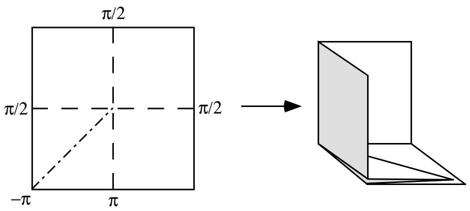

Figure 2.1: a 3-D origami.

The focus of this paper is origamis which fold flat. In such origamis we think of each fold as being at π or $-\pi$ radians, but in actuality this does not happen, as it would result in a mapping into $\mathbf{R}^3$ which is not 1-1. Thus we say

Definition: A *flat origami* is an origami such that if we take the limit as each valley crease is folded to π radians and as each mountain crease is folded to $-\pi$ radians, then the origami is still 1-1 as we take this limit.

The main requirement, of course, is that we want to fold all creases flat without having the paper intersect itself. The fold in Figure 2.1 does not have this property, but the crane shown in Figure 1.2 does.

Since in a flat origami the angle of each fold is determined by whether it is a mountain or a valley, we may describe flat origamis by a pair $(\mathcal{C}, f)$ where $\mathcal{C}$ is the crease pattern and $f : \mathcal{C} \to \pm 1$ defines the parity of the creases.

3 Local Properties of a Flat Origami

We concentrate on local properties of a flat origami. I.e., we explore what can be said about the crease-pattern surrounding an individual vertex in a flat origami. By "vertex" we mean a point where crease lines meet in the *interior* of the square.

In this regard, consider the unit disk D to be our folding medium and our flat origami $(\mathcal{C}, f)$ to have radial creases with its only vertex at the origin (see Figure 3.1). We first look at a result due to Jun Maekawa (see [3]) relating the mountain and valley creases about the vertex.

Maekawa's Theorem 3.1 *Let M denote the number of mountain creases in $\mathcal{C}$ and V be the number of valley creases. Then $M - V = \pm 2$.*

The proof we will present was conceived by high school student Jan Siwanowicz during an origami-math mini course that the author taught at the 1993 Hampshire College Summer Studies in Mathematics.

Proof: (Jan Siwanowicz) Consider the unit disk D after it is folded. The one vertex in the crease-pattern will then be a point in this folded figure. If we cut this point off, a polygonal cross-section of the folded paper is revealed (see Figure 3.1). Notice that since this is a flat origami all of the interior angles of this polygon will be either 0 or 2π radians.

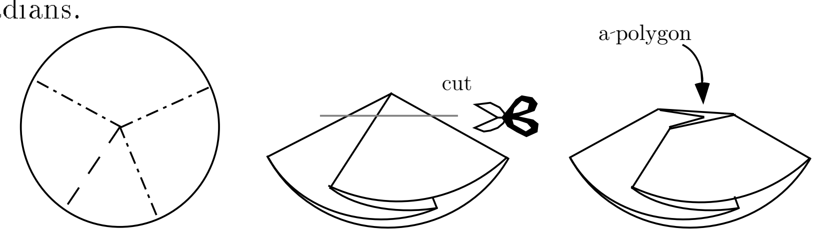

Figure 3.1

Now, the mountain folds of $\mathcal{C}$ correspond to the 0 angles of the polygon, and the valley folds correspond to the 2π angles. If we let n be the number of creases in $\mathcal{C}$ (which equals the number of sides of our polygon), then the sum of the interior angles of the polygon equals $(n-2)\pi$. But since $n = M + V$ as well, we have

$$0M + 2\pi V = (M + V - 2)\pi \Rightarrow M - V = 2.$$

This was assuming that our vertex pointed "up" when folded. If it had pointed "down," then the above analysis would have given $V - M = 2$. Thus $M - V = \pm 2$. $\square$

Corollary 3.1 *The number of creases in $\mathcal{C}$ is even.*

Proof: If n is the number of creases in $\mathcal{C}$ and $M - V = 2$ then $n = M + V = M - V + 2V = 2(1 + V)$. A similar trick works if $V - M = 2$. $\square$

Corollary 3.2 *If we think of an origami crease-pattern as a graph, then every flat origami crease-pattern is 2 face-colorable.*

Proof: The previous corollary gives us that every vertex is of even degree with the exception of possible vertices along the sides of the square. There are an even number of such side-vertices, however, so we may create a new vertex *outside* of the square and connect this vertex to each side-vertex of odd degree. The resulting graph is Eulerian, thus 2 face-colorable, which gives us the desired coloring for the crease pattern. $\square$

This last corollary was first discovered by Meguro of Japan. It is interesting because it demonstrates the parity of the faces in the crease pattern when folded flat. If we color the faces red and blue, then the final flat folded figure will have all red faces facing in one direction, and the blue faces in the other. The reader is encouraged to fold any flat origami (such as the traditional crane in Figure 1.2), unfold it and actually 2-color the faces. Then re-fold the figure to see this face parity in action. It makes a great party trick.

We now examine properties, which were first demonstrated by Kawasaki in [5] and [6], of the angles between creases in our flat origami $\mathcal{C}$. If $\mathcal{C}$ has $2n$ creases, let us denote the angles between the creases by $\alpha_1, \alpha_2, ..., \alpha_{2n}$.

Theorem 3.2 *The sum of the alternate angles about the vertex in $\mathcal{C}$ is π.*

Proof: Let γ be a simple closed curve in D around the vertex of $\mathcal{C}$. If we fold D using the creases in $\mathcal{C}$ and follow the trace of γ on this folded shape, we see that every time γ meets a crease line we must change our direction as we proceed along the trace of γ. Eventually we come back to where we started on the trace of γ, and this implies that

$$\alpha_1 - \alpha_2 + \alpha_3 + \cdots - \alpha_{2n} = 0 \tag{1}$$

(see Figure 3.2). But we also have that $\alpha_1 + \alpha_2 + \cdots + \alpha_{2n} = 2\pi$. Adding this to (1) and simplifying yields $\alpha_1 + \alpha_3 + \cdots + \alpha_{2n-1} = \pi$. This then implies that $\alpha_2 + \alpha_4 + \cdots + \alpha_{2n} = \pi$ as well. $\square$

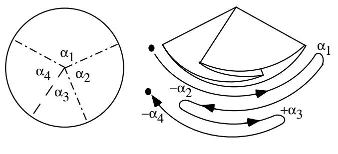

Figure 3.2: Equation (1) in action.

Not only is the result of Theorem 3.2 necessary for a flat origami, it is sufficient as well. This surprising fact implies that whether or not a set of lines can be used to make a (locally) flat origami depends only on the angles about each vertex, not on the arrangement of the mountain or valley folds.

With this in mind, suppose that $\mathcal{C} = \{l_1, ..., l_{2n}\}$ is a collection of radial lines in the unit disk D. We say that $\mathcal{C}$ *generates a flat origami* if there exists a function $f : \mathcal{C} \to \{\pm 1\}$ such that $(\mathcal{C}, f)$ is a flat origami.

Theorem 3.3 *Let $\mathcal{C} = \{l_1, ..., l_{2n}\}$ be a collection of radial lines in the unit disk D, and denote the angles between these lines by $\alpha_1, \alpha_2, ..., \alpha_{2n}$. Suppose further that $\alpha_1 + \alpha_3 + \cdots + \alpha_{2n-1} = \alpha_2 + \alpha_4 + \cdots + \alpha_{2n} = \pi$. Then $\mathcal{C}$ generates a flat origami.*

Proof: The conditions of the Theorem insures us that equation (1) holds, and this tells us that, at least as far as the angles are concerned, the disk D can be folded flat using the creases in $\mathcal{C}$. All that remains to be shown is that a suitable arrangement of mountain and valley assignments for the creases in $\mathcal{C}$ exists (i.e., the function f) which will make a legitamate flat origami.

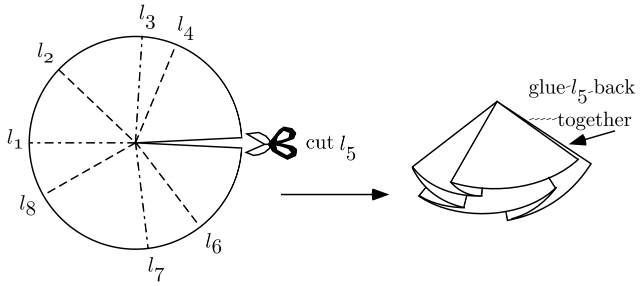

Figure 3.3: one example of how to assign the mountain and valley creases.

To assign the mountain and valley creases, let l_1 be a valley, then let l_2 and l_{2n} be mountains, then let l_3 and l_{2n-1} be valleys, then the next two mountains, etc. Then *cut* along crease l_{n+1} and fold the disk into what paperfolders call an "accordian pleat" (see Figure 3.3).

Since equation (1) holds, we know that the two loose ends that were once l_{n+1} will line up when the other creases are folded. If we're lucky there won't be any other layers of paper in between these loose ends, and then we may glue them back together to complete the fold (as shown in Figure 3.3).

If we're unlucky and there are layers of paper between the loose ends, then a strategy like the one shown in Figure 3.4 must be employed. The idea is to reverse the crease which projects to the outermost left or right in the folded paper. Doing this will place the loose ends together, which may then be glued. □

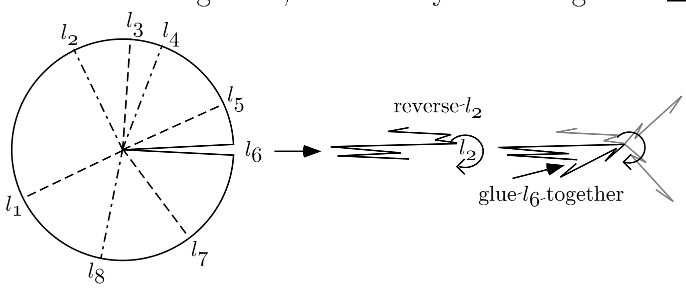

Figure 3.4: another example of how to assign creases.

Notice that the specific crease assignment of l_{n+1} will depend on whether n is odd or even. We shall refer to this necessary and sufficient condition for a flat origami to locally fold flat as the "180° condition."

4 Problems with Extending Globally

4.1 $M - V = \pm 2$

Theorem 3.1 does not extend to flat origamis with more than one vertex (for example, in the origami shown in Figure 1.2, $M - V = 15$). However, $M - V$ does seem to measure certain properties of a flat origami.

If we consider the crease pattern of an arbitrary flat origami, then *locally* $M - V = \pm 2$ will hold for each vertex in the interior of the square (places where crease lines intersect on the edges of the square are not considered to be vertices). In fact, $M - V = 2$ if the vertex is pointing "up" and $M - V = -2$ if the vertex points "down" after it is folded. This suggests that *globally* we have

$$M - V = 2(\#\text{ of up vertices}) - 2(\#\text{ of down vertices}).$$

But this is incorrect, since in counting the number of up and down vertices we have perhaps counted some creases twice. Thus we define a crease line to be an **interior crease** if its endpoints lie in the interior of the square. We then obtain the following result:

Proposition 4.1 *Let (C, f) be a flat origami and let M and V denote the number of mountain and valley creases in (C, f), respectively. Then*

$$\begin{aligned} M - V &= 2(\#\textit{ of up vertices}) - 2(\#\textit{ of down vertices}) \\ &\quad -(\#\textit{ of interior mountain creases}) + (\#\textit{ of interior valley creases}). \end{aligned}$$

The only thing that could thwart this proposition is a crease line which does not intersect any other creases in the interior of the square. For example, an origami consisting of a single mountain fold would have $M - V = 1$, contradicting our proposition.

To patch these cases up, we must consider such a crease line to be actually *two* creases with a vertex placed anywhere along the line. Thus, our flat origami with just a single mountain fold actually has one vertex and two crease lines, both mountains. Then $M - V = 2$, which agrees with the Proposition 4.1 (and Theorem 3.1).

Question: Can $M - V$ be used as an "origami characteristic" in any way? We already know that if (C, f) is a flat origami with only one vertex, then $M - V = \pm 2$ (although the converse is not true). Any other sweeping characterizations like this seem elusive.

4.2 The 180° condition

Theorems 3.2 and 3.3 do not extend naturally to flat origamis with multiple vetricies either. Unfoldable crease-patterns which are locally foldable do exist. One example is shown if Figure 4.2. To see why, we provide a proposition from [5]. In the following, let $\alpha_1, \alpha_2, ..., \alpha_{2n}$ denote the angles between the crease lines in a flat origami with only one vertex.

Proposition 4.2 *For any flat origami $(\mathcal{C}, f)$ defined on the unit disk D (whose only vertex is at the origin), if $\alpha_i < \alpha_{i-1}$ and $\alpha_i < \alpha_{i+1}$, then $f(l_i) = -f(l_{i+1})$.*

To see why this is true, see Figure 4.1. If $f(l_i) = f(l_{i+1})$, then folding these two creases would cause either the α_{i-1} or the α_{i+1} flap to rip through crease line l_{i+1} or l_{i-1}, respectively.

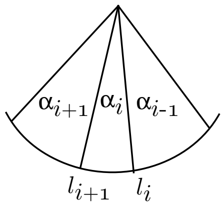

Figure 4.1

Now consider the crease-pattern in Figure 4.2. Each vertex folds flat (two right angles alternate about each vertex) and each fit the requirements of Corollary 4.1. Thus if the crease line l_1 were a mountain, l_2 would have to be a valley. But this would imply that l_3 must be a mountain, and we get a contradiction with Corollary 4.1 on the lower left-hand vertex (l_1 and l_3 cannot both be mountains). Thus this crease pattern is unfoldable.

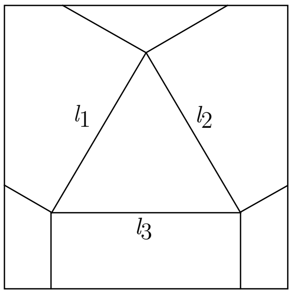

Figure 4.2: an impossible fold.

However, notice what we are doing. In trying to fold such a crease-pattern, we assign a parity to the crease lines, mountain or valley. This can be thought of as a type of edge coloring, but not of the normal type where adjacent edges are given different colors. Rather, we assign different colors to the crease lines if they *must* be of different mountain/valley parity. In lieu of this, we make the following definition.

Definition: Given a set $\mathcal{C}$ of crease lines (which may or may not generate a flat origami) we construct the **origami line graph** G as follows:

(1) The vertices of G are the crease lines in $\mathcal{C}$.

(2) Two crease lines l_i and l_j form an edge in G if and only if

(i) l_i and l_j are adjacent in $\mathcal{C}$, and

(ii) $f(l_i) = -f(l_j)$ must be true if $\mathcal{C}$ is to generate a flat origami $(\mathcal{C}, f)$.

Condition (2)(ii) simply means that l_i and l_j can neither both be mountains nor both be valleys. Figure 4.3 shows what the origami line graph for the crease pattern in Figure 4.2 would be. Notice that it is a subgraph of the conventional line graph. Also notice that it is not 2 vertex-colorable, which again illustrates why this crease-pattern is unfoldable.

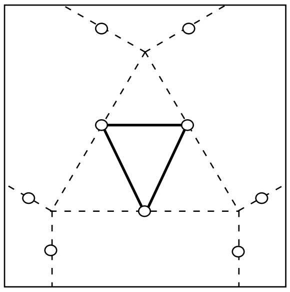

Figure 4.3: an origami line graph

Note that the origami line graph might not be easy to construct. Indeed, Figure 4.4 shows another cerase pattern (a simplification of one found in [5]) which can be locally folded flat but is impossible to fold, as the shown mountain/valley assignments illustrate.

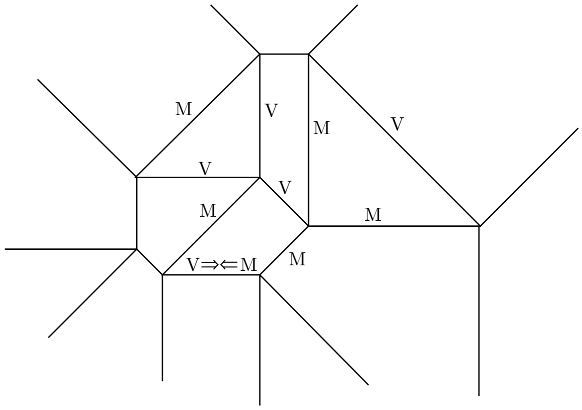

Figure 4.4: another impossible fold

Constructing the origami line graph of this crease-pattern and seeing why it is not 2 colorable is possible, but not as straightforward as the previous example. The reader is encouraged to try.

However, we are tempted to make a conjecture.

Conjecture 4.1 *A collection $\mathcal{C}$ of crease lines in the square (possibly with more than one vertex) can generate a flat origami if and only if*

(i) each vertex in $\mathcal{C}$ satisfies the 180° *condition and*

(ii) the origami line graph of $\mathcal{C}$ is 2 vertex-colorable.

A reader who hasn't had much experience in folding paper may think that the above discussion constitutes a near proof of this conjecture. But alas, this is not the case. The origami line graph provides us with a way for checking the existence of a valid mountain/valley assignment, but doesn't guarantee that the 1-1 property will not be broken. And indeed it might. Figure 4.5 displays a crease pattern which satisfies the conditions of the conjecture (indeed, the origami line graph is empty here!), but is unfoldable.

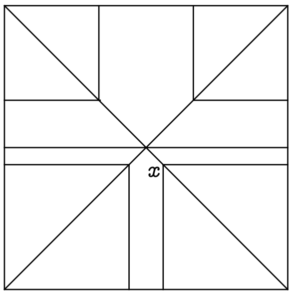

Figure 4.5: a much more subtle impossible fold

The reader is encouraged to experiment with this crease-pattern to see exactly why all the creases cannot be folded without ripping the paper. One way to patch things up might be to place weights on the creases to indicate their Euclidean lengths. But the possible combinations in which the folded flaps can be layered in this example is quite involved. Further, all we would have to do to make this foldable would be to, say, move the vertex x a little bit closer to the lower right corner. Examination of this example makes one suspect that there may not be an easy condition that we could tack onto the conjecture to make it true.

Addendum: At the time of publication it has come to the attention of the author that some of the results in this paper have been independently established by others. For example, the concept of the origami line graph seems to have been discovered by Jacques Justin of France. Thus, since the subject of origami mathematics is only now beginning to bloom internationally, this paper should be thought of as primarily an attempt to communicate these results to the English-speaking mathematical community.

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