Proof of a Conjecture of Technical Origami.

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Abstract. It has been conjectured that if a pattern of non-overlapping circles , centres (x_i,y_i), radii r_i, are laid out on a sheet of paper, then there is *always* a way of folding the paper into a flat origami with flaps of length r_i, whose tips correspond to the points (x_i,y_i). This conjecture is used in much of modern origami design. This paper is intended to provide some better justification for a belief in this proposition, although the current work falls just short of being a proof of the entire conjecture.

We start our discussion apparently some way away from the original problem, looking at tiling the plane with triangles. The connection to the problem in hand will become clear towards the end of the paper.

Consider the problem of covering a finite region of the plane, *R*, with triangular tiles. In particular, the tiling process, adding one tile at a time, to a space beside one or more already placed tiles, with the vertices of adjacent triangles meeting. The tiles are of varying sizes and angles. Assume that we have already decided where to place these tiles; the problem we will consider here is the order in which they are placed.

Theorem 1. It is always possible to tile in the manner described so that a tile never has to be placed in a space where all three adjacent triangles have already been tiled.

Proof. Suppose this is not possible. Then it is possible to have a space or inclusion inside a tiled region. From some tile in this region, draw a line to the region of the plane outside *R*. Remove all of the tiles passed through by this line, calling the region so revealed (together with the original inclusion) *S*, and the outermost removed tile t_0.

Define a distance d_i for each space t_i in *S* by

$$d_i = \text{min over all } p(t_i,t_0) \text{ of number of triangles passed through.}$$

Where $p(t_i,t_0)$ is a path within *S* which starts at t_i and ends at t_0. Now start retiling S, by placing a tile in the space for which di is largest. After placing each tile, recalculate di. It is easily seen that, firstly, we are indeed placing tiles beside previously placed tiles in this algorithm, and secondly, that we cannot create a new inclusion this way. Thus, we have given an algorithm which offers an alternative way of tiling a region with an inclusion so that it need not have an inclusion, which is the required result.

Lemma 1. When tiling in the above manner, if a tile is placed adjacent to two others, it is the last tile that is to be placed at the vertex where these two meet.

Proof. Suppose otherwise. Then one of the situations in diagram 1 must occur.

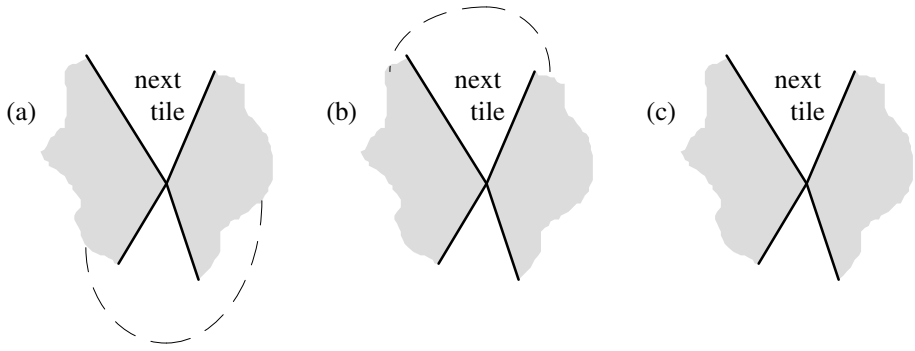


Diagram 1.

In (a), the two tiled sections at he vertex are connected around the 'other' gap at this vertex. This cannot happen, since by theorem 1 we can tile without inclusions. Similarly in (b), where the regions

are connected around the place where this tile is to be placed.. In (c) the two tiled regions are disjoint. This also cannot happen, since we tile adjacent to previously tiled regions only, and so there must be a path between every placed tile that only passes through tiled regions.

Now we introduce the tiles we are actually going to use, and demonstrate their connection with the original conjecture. All of the tiles have the same basic construction of crease lines laid out on them, as shown in diagram 2. (Origamians may recognise these as being similar to the molecules of Meguro and Husimi). We impose a further condition on the tiles, that when two tiles are adjacent, the perpendiculars of their sides meet.

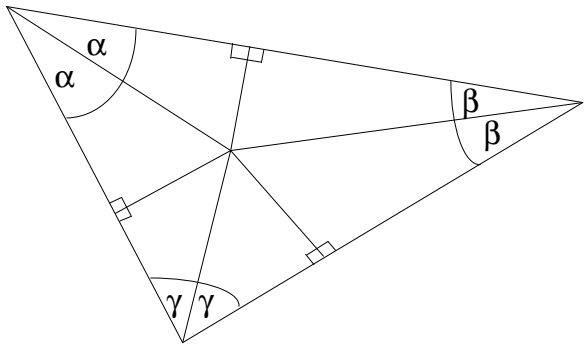
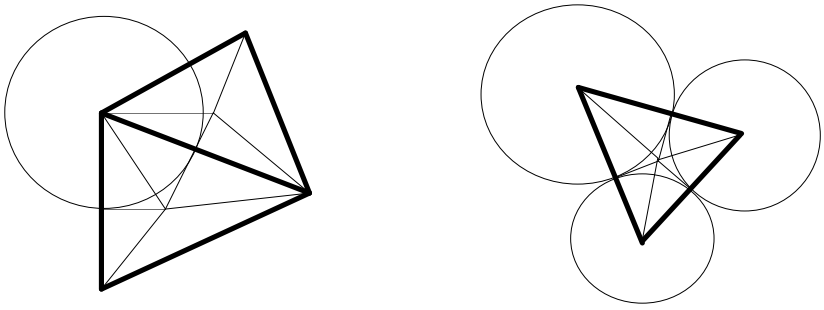


Diagram 2.

This condition on perpendiculars meeting can easily be shown to mean that, if we look at the tiles around some vertex, the perpendiculars which join pairs of these tiles form a polygon whose sides are tangents to a circle centred at the vertex, with its radius being the distance from any of the points where the perpendiculars meet to the vertex. (diagram 3(a)). Furthermore, we can construct the tiling in the other direction: drawing three circles so that each touches the other two, we can construct a tile of the form described above by connecting their centres, drawing tangents from the point where the circles meet to the point where all three tangents meet, and finally drawing lines from this point to the three centres (diagram 3(b)).



(a) Joined tiles. (b) Tiles from packed circles.

Diagram 3: Tiles related to underlying circle design.

Thus, if the pattern of circles in our origami design is of the form where, if two circles are in contact, there is always another with which they are both in contact, then we can construct a tiling with triangles of the above form of the design.

A first condition that must be satisfied by a crease pattern is that Kawaskai's theorem holds at every vertex. That is that given the angles around the vertex $\alpha_1..\alpha 2_n$, then:

$$\Sigma_{i=1..2n} (-1)^n \alpha_i = \Sigma_{i=1..n} \alpha_{2i} - \alpha_{2i-1} = 0.$$

It is immediately obvious that these tiles satisfy the second form of this equation at each corner, and will do so no matter how many of them are tiled around a vertex. Similarly for the vertices that occur on the sides of the triangles: since all of the angles are the same, ie $\pi/_2$, and there are an even number of them, they will automatically satisfy the above equation. Finally, a little thought will

show that there are pairs of equal adjacent angles at the vertex in the centre of each tile, so that Kawaskai's theorem is also satisfied there.

Now we assign creases to the tiles. We limit the possible assignments of creases to 5, which fall into two distinct sets, and are shown in diagram 3. They are designed to ensure that I can satisfy Maekwa's theorem, that:

$$\text{Mountain folds - Valley folds} = {}^{+}/_{-}\ 2$$

At every vertex simultaneously.

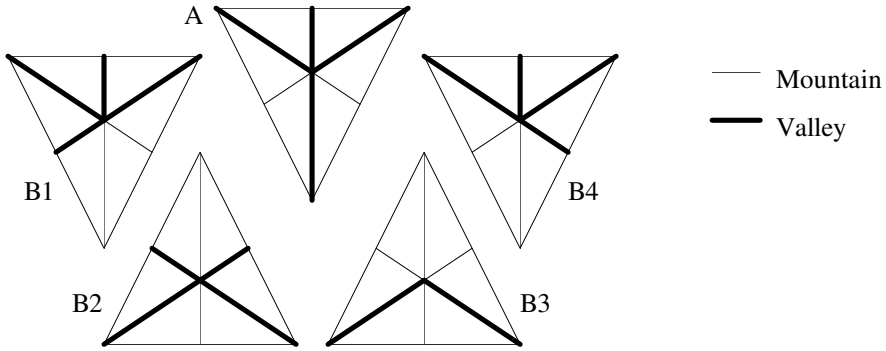


Diagram 4. The Complete Set of Tiles.

We proceed to tile as follows. When placing a tile adjacent to only one other previously placed tile, we use a tile of type A. When placing the final tile at each vertex, we use a tile of type B. It is a simple, if somewhat laborious, task to show that this can be done, by enumerating all possible A-A pairs and A-B-A triples that can occur in this method. It is somewhat quicker to note that the possibilities are oll of the form shown in diagrams 5 and 6. Note that I can always place the tile so that where perpendiculars meet, Maekwa's theorem holds, and that it also holds at the verices at the centres of each tile.

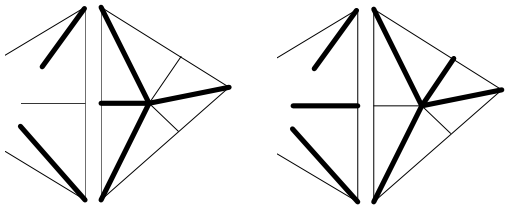


Diagram 5: All possible situations tiling type A.

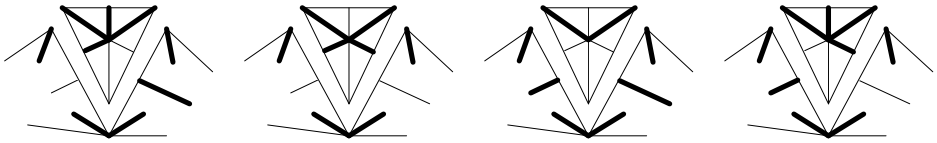


Diagram 6: All possible situations tiling type B.

It merely remains to show that Maekwa's theorem holds at every vertex where the tiles meet. After a little thought, we see that tiling completely around a vertex with tiles of type A would give M-V=0. Replacing one tile of type A with a tile of type B so that a valley fold is removed, and a mountain fold is added, gives us:

$$M' - V' = M + 1 - (V - 1) = M - V + 2 = 2$$

Which is the required result. This is enforced by the algorithm described above for placing tiles of each type.

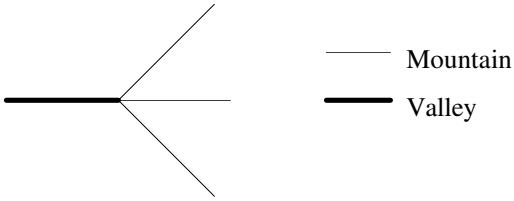
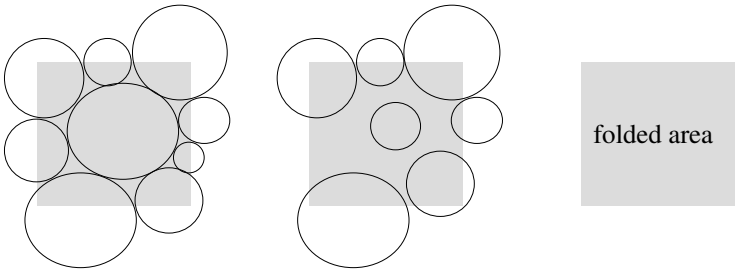


Diagram 7: An un-foldable origami.

It is well known that a origami crease pattern can be locally foldable flat and satisfy Maekwa's theorem everywhere, and still not be foldable - the simplest counterexample is shown in diagram 7. Hence this does not prove that an origami constructed from a packed circle design can be folded. However, it does provide at least some justification for a belief in this possibility. Finally, I have only shown the result when the design is packed with circles, as in diagram 8(a). To show that the method works for general tilings of circles, (diagram 8(b)) it is required to show that a region bounded by an arbitrary number of circles of varying sizes can be tiled by a finite number of circles in a packed way. This question remains open for future investigation.



(a) packed. (b) normal.

Diagram 8. Circle Designs.