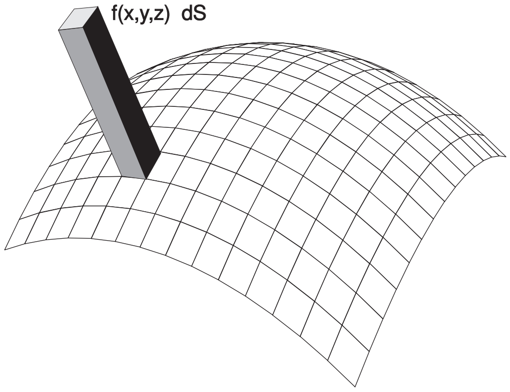
12-29-1993

Surface integrals.

If S is a surface and $f(x, y, z)$ is a function, the **scalar surface integral** of f over S is

$$\iint_S f\, dS.$$

Imagine placing a grid on the surface. dS represents the area of a small parallelogram in the grid. At a point (x, y, z), build a "box" on the grid at (x, y, z) whose height is $f(x, y, z)$. The volume of the box will be product of the height ($f(x, y, z)$) and the parallelogram area (dS), i.e. $f(x, y, z)dS$.



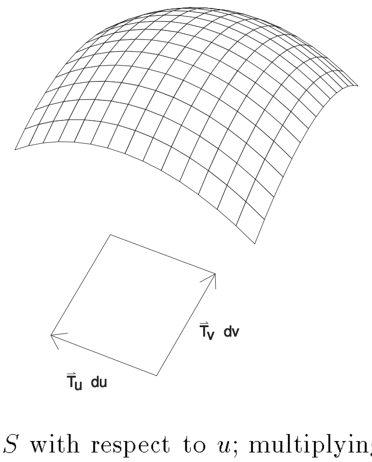
Under this heuristic interpretation, the scalar surface integral represents the total volume of all the "boxes" built in this way on the surface.

It is fairly clear how to deal with $f(x, y, z)$, but what about dS? Suppose the surface is parametrized:

$$(x, y, z) = \Phi(u, v).$$

That is, each of x, y, and z is expressed in terms of parameters u and v. Fix v and vary u. This gives a curve on the surface, whose tangent vector I'll denote by $\overrightarrow{T_u}$. Likewise, fixing u and varying v produces a curve on the surface whose tangent vector I'll denote by $\overrightarrow{T_v}$.

A small part of the surface grid pictured above can be thought of as a parallelogram whose sides are given by the vectors $\overrightarrow{T_u}\, du$ and $\overrightarrow{T_v}\, dv$.



$\overrightarrow{T_u}$ gives the "rate of change" of S with respect to u; multiplying by a small change du in u gives the approximate change $\overrightarrow{T_u}\, du$ in S. Likewise, $\overrightarrow{T_v}\, dv$ gives the change in S produced by a small change in v.

The area of the little parallelogram is the length of the cross product of its sides:

$$dS = |\overrightarrow{T_u} \times \overrightarrow{T_v}|\, du\, dv.$$

$\overrightarrow{T_u} \times \overrightarrow{T_v}$ is the **normal vector** to the surface, since each factor is tangent to the surface. Thus, to compute a scalar surface integral, use the following formula:

$$\int\int_S f\, dS = \int\int_D f\left(x(u,v), y(u,v), z(u,v)\right) |\overrightarrow{T_u} \times \overrightarrow{T_v}|\, du\, dv.$$

- The region D which gives the bounds for the double integral is given by the ranges for the parameters u and v.
- "$f\left(x(u,v), y(u,v), z(u,v)\right)$" means that you should use the parametric equations for the surface to convert f from x, y, and z to u and v.
- You can compute the normal vector using

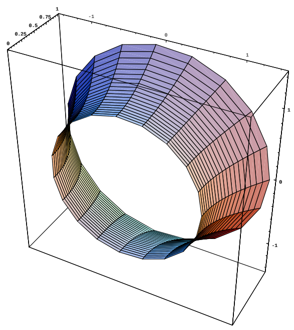
$$\overrightarrow{T_u} \times \overrightarrow{T_v} = \left| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{matrix} \right|.$$

Example. If $f = 1$ in $\int\int_S f dS$, the surface integral of f is just the **area** of the surface:

$$A = \int\int_S dS.$$

For example, consider the surface given by

$$x = \sec u \cos v, \quad y = \tan u, \quad z = \sec u \sin v.$$



It is the surface of revolution generated by revolving the curve $x = \sec u$, $y = \tan u$, about the y-axis. In x-y-z coordinates, it is $x^2 - y^2 + z^2 = 1$, as you can verify by substituting for x, y and z.

I'll find the area of the part of the surface corresponding to the parameter ranges $0 \le u \le \dfrac{\pi}{4}$, $0 \le v \le 2\pi$.

Compute the normal $|\overrightarrow{T_u} \times \overrightarrow{T_v}|$:

$$\overrightarrow{T_u} \times \overrightarrow{T_v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \sec u \tan u \cos v & (\sec u)^2 & \sec u \tan u \sin v \\ -\sec u \sin v & 0 & \sec u \cos v \end{vmatrix} = \left\langle (\sec u)^3 \cos v, -(\sec u)^2 \tan u, (\sec u)^3 \sin v \right\rangle.$$

Now find the length of the normal:

$$|\overrightarrow{T_u} \times \overrightarrow{T_v}| = \left((\sec u)^6 (\cos v)^2 + (\sec u)^4 (\tan u)^2 + (\sec u)^6 (\sin v)^2\right)^{1/2} =$$

$$\left((\sec u)^6 + (\sec u)^4 (\tan u)^2\right)^{1/2} = (\sec u)^2 \sqrt{(\sec u)^2 + (\tan u)^2}.$$

The area of the surface is

$$\int_0^{2\pi} \int_0^{\pi/4} (\sec u)^2 \sqrt{(\sec u)^2 + (\tan u)^2}\, du\, dv = 2\pi \int_0^{\frac{\pi}{4}} (\sec u)^2 \sqrt{1 + 2(\tan u)^2}\, du =$$

$$\frac{\sqrt{3}}{2} + \frac{\ln(\sqrt{2} + \sqrt{3})}{2\sqrt{2}} \approx 1.27127.$$

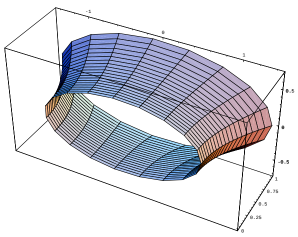
(I computed the integral using the identity $(\sec u)^2 = 1 + (\tan u)^2$, and then the substitution $w = \tan u$, $dw = (\sec u)^2\, du$.) $\square$

Now consider a vector field $\vec{F}$ in space, and let S be a surface. If you think of F as the velocity field of a fluid or gas and the surface S as a membrane, it is natural to ask "how much" fluid or gas passes through the membrane per unit time. This rate is called the **flux** of $\vec{F}$ through S, and is given by the **vector surface integral**

$$\int\int_S \vec{F} \cdot d\vec{S} = \int\int_D \vec{F} \cdot (\overrightarrow{T_u} \times \overrightarrow{T_v})\, du\, dv.$$

(I'm assuming that the surface is parametrized by $(x, y, z) = \Phi(u, v)$.)

Examples. 1. (a) Parametrize the elliptic hyperboloid $x^2 - y^2 + 4z^2 = 1$.



In the last example, I noted that the surface $x^2 - y^2 + z^2 = 1$ may be parametrized by

$$x = \sec u \cos v, \quad y = \tan u, \quad z = \sec u \sin v.$$

The surface in this example is the same except for a "4" in the z term. So I can use the same parametrization, provided that I insert a factor of $\frac{1}{2}$ in the z term to cancel the "4":

$$x = \sec u \cos v, \quad y = \tan u, \quad z = \frac{1}{2} \sec u \sin v.$$

Check:

$$x^2 - y^2 + 4z^2 = (\sec u)^2(\cos v)^2 - (\tan u)^2 + 4 \cdot \frac{1}{4}(\sec u)^2(\sin v)^2 = (\sec u)^2 - (\tan u)^2 = 1. \square$$

(b) Compute the flux of the radial vector field $\vec{F} = \langle x, y, z\rangle$ outward through the part of the surface in (a) determined by the parameter ranges $-\frac{\pi}{4} \le u \le \frac{\pi}{4}$, $0 \le v \le 2\pi$.

The normal is

$$\overrightarrow{T_u} \times \overrightarrow{T_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sec u \tan u \cos v & (\sec u)^2 & \frac{1}{2}\sec u \tan u \sin v \\ -\sec u \sin v & 0 & \frac{1}{2}\sec u \cos v \end{vmatrix} =$$

$$\left\langle \frac{1}{2}(\sec u)^3 \cos v, -\frac{1}{2}(\sec u)^2 \tan u, (\sec u)^3 \sin v \right\rangle.$$

For the given ranges of u and v, the x and z components of the normal are *positive*, so the normal points *out of* the hyperboloid. (If the normal had turned out to point *inward*, I'd have simply multiplied it by -1 to get the outward normal.)

Next, write the field in terms of u and v:

$$\vec{F} = \left\langle \sec u \cos v, \tan u, \frac{1}{2}\sec u \sin v \right\rangle.$$

Therefore,

$$\vec{F} \cdot (\overrightarrow{T_u} \times \overrightarrow{T_v}) = \frac{1}{2}(\sec u)^2.$$

Hence, the flux is

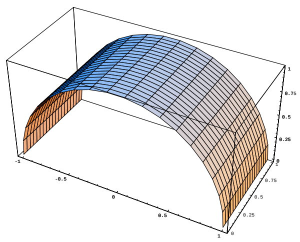
$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_{-\pi/4}^{\pi/4} \frac{1}{2}(\sec u)^2 \, du \, dv = 2\pi \approx 6.28319. \square$$

2. If the surface is given as the graph of a function $z = f(x, y)$, a normal is given by

$$\vec{N} = \pm \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle.$$

(As noted in the remarks at the end of the last example, you must decide whether to use $\vec{N}$ or $-\vec{N}$ based on the wording of the problem.)

For example, I'll find the flux of $\vec{F} = \langle 2x, 0, z\rangle$ out of the cylinder $z = \sqrt{1 - x^2}$, where $-1 \le x \le 1$ and $0 \le y \le 1$.



The normal is

$$\left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right\rangle = \left\langle \frac{x}{\sqrt{1-x^2}}, 0, 1\right\rangle.$$

This cylinder is an "x-z" cylinder, with the y-axis as its axis. So the *inward* normal will have *negative* x and z components, while the *outward* normal will have *positive* x and z components. The normal above has positive x and z components, so it's the right one.

Next,

$$\vec{F}\cdot\vec{N} = \langle 2x, 0, z\rangle \cdot \left\langle \frac{x}{\sqrt{1-x^2}}, 0, 1\right\rangle = \frac{1+x^2}{\sqrt{1-x^2}}.$$

Hence, the flux is

$$\iint_S \vec{F}\cdot d\vec{S} = \int_0^1 \int_{-1}^1 \frac{1+x^2}{\sqrt{1-x^2}}\, dx\, dy = \frac{3\pi}{2} \approx 4.71238. \square$$

Problems on Surface Integrals.

1. Compute

$$\iint_S (x+4y+z)\, dS,$$

where S is the triangle with vertices $P(1,1,2)$, $Q(2,3,1)$, $R(-1,2,0)$.

2. Let S be the surface $z = 4 - \sqrt{x^2+y^2}$ for $z \geq 0$.

(a) Find the area of S.

(b) Suppose the surface is a thin sheet of metal whose density is $\delta(x,y,z) = x^2$. Find the mass of the sheet of metal.

3. (a) Let S be the surface $z = 4 - \sqrt{x^2+y^2}$ for $z \geq 0$. Parametrize S as a surface of revolution.

(b) Compute the flux of $\vec{F} = \langle x, y, 2\rangle$ upward through the surface in (a).

Solutions to the Problems on Surface Integrals.

1. Compute

$$\iint_S (x+4y+z)\, dS,$$

where S is the triangle with vertices $P(1,1,2)$, $Q(2,3,1)$, $R(-1,2,0)$.

$$\overrightarrow{PQ} = \langle 1, 2, -1\rangle, \quad \overrightarrow{PR} = \langle -2, 1, -2\rangle.$$

Therefore, the triangle may be parametrized by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} u-2v+1 \\ 2u+v+1 \\ -u-2v+2 \end{bmatrix}.$$

In component form, this is

$$x = u-2v+1, \quad y = 2u+v+1, \quad z = -u-2v+2, \qquad 0 \leq u \leq 1, \quad 0 \leq v \leq 1-u.$$

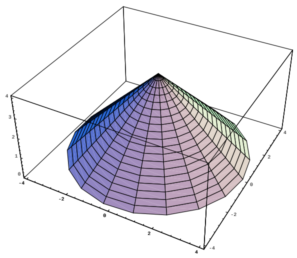
The normal vector is

$$\vec{N} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 1 & -2 \end{vmatrix} = \langle -3, 4, 5 \rangle.$$

Therefore, $|\vec{N}| = \sqrt{50}$. Since $x + 4y + z = 8u + 7$,

$$\iint_S (x + 4y + z)\, dS = \int_0^1 \int_0^{1-u} (8u + 7)\sqrt{50}\, dv\, du = \sqrt{50} \int_0^1 (8u + 7)(1 - u)\, du = \frac{29\sqrt{50}}{6} \approx 34.17683. \square$$

2. Let S be the surface $z = 4 - \sqrt{x^2 + y^2}$ for $z \geq 0$.



(a) Find the area of S.

The normal vector to $z = 4 - \sqrt{x^2 + y^2}$ is

$$\vec{N} = \langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \rangle = \langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, 1 \rangle.$$

Therefore, $|\vec{N}| = \sqrt{2}$.
The projection of the surface in the x-y-plane is the disk $x^2 + y^2 \leq 16$. In polar, it is

$$0 \leq r \leq 4, \quad 0 \leq \theta \leq 2\pi.$$

The area is

$$\iint_S dS = \int_0^4 \int_0^{2\pi} \sqrt{2} r\, d\theta\, dr = 16\sqrt{2}\pi \approx 71.08613. \square$$

(b) Suppose the surface is a thin sheet of metal whose density is $\delta(x, y, z) = x^2$. Find the mass of the sheet of metal.

The mass is given by the surface integral

$$\iint_S \delta(x, y, z)\, dS.$$

Everything is the same as it was in part (a). I only need to insert the density δ after converting it to polar:

$$\delta = r^2(\cos\theta)^2.$$

The mass is

$$\iint_S \delta(x, y, z)\, dS = \int_0^4 \int_0^{2\pi} r^2 (\cos\theta)^2 \sqrt{2} r \, d\theta \, dr = 64\pi\sqrt{2} \approx 284.34451. \square$$

3. (a) Let S be the surface $z = 4 - \sqrt{x^2 + y^2}$ for $z \ge 0$. Parametrize S as a surface of revolution.

As you can see from the picture in the last problem, the surface is a cone whose axis is the z-axis. The curve being revolved should lie in either the x-z or y-z-plane. I'll use the x-z-plane: this is $y = 0$. The equation becomes $z = 4 - x$ for $0 \le z \le 4$. I can parametrize this *curve* by

$$x = u, \quad z = 4 - u, \quad 0 \le u \le 4.$$

Therefore, the *surface of revolution* may be parametrized by

$$x = u\cos v, \quad y = u\sin v, \quad z = 4 - u, \qquad 0 \le u \le 4, \quad 0 \le v \le 2\pi. \square$$

(b) Compute the flux of $\vec{F} = \langle x, y, 2 \rangle$ upward through the surface in (a).

The normal is

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & -1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle u\cos v, u\sin v, u \rangle.$$

Write $\vec{F}$ in terms of u and v:

$$\vec{F} = \langle u\cos v, u\sin v, 2 \rangle.$$

Then

$$\vec{F} \cdot \vec{N} = u^2 + 2u.$$

The flux is

$$\iint_S \vec{F} \cdot \overrightarrow{dS} = \int_0^{2\pi} \int_0^4 (u^2 + 2u)\, du\, dv = \frac{224\pi}{3} \approx 234.57225. \square$$