Math 123
2-28-1994

Partial fractions.

Partial fractions is the opposite of adding fractions over a common denominator. It applies to integrals of the form

$$\int \frac{P(x)}{Q(x)}\,dx,$$

where $P(x)$ and $Q(x)$ are polynomials. The idea is to break $\dfrac{P(x)}{Q(x)}$ into a sum of smaller terms which are easier to integrate.

(A function of the form $\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials, is called a **rational function**.)

I'll explain how partial fractions works by looking at some examples. The procedure is a bit long, and requires a substantial amount of algebra. Therefore, before using partial fractions, you should be sure that there isn't an easier way to do the integral.

Let's assume, then, that we're looking at an integral of the form

$$\int \frac{P(x)}{Q(x)}\,dx,$$

where $P(x)$ and $Q(x)$ are polynomials.

Recall that the **degree** of a polynomial is the highest power of the variable that occurs in it. Nonzero constants have degree 0; by convention, 0 has degree $-\infty$.

Step 1. If the degree of the top is greater than or equal to the degree of the bottom, divide the bottom into the top.

Examples.

1. $\displaystyle\int \frac{2x^3-5x^2-x+5}{x^2-1}\,dx$

The top has degree 3 while the bottom has degree 2. Divide the bottom into the top:

$$\frac{2x^3-5x^2-x+5}{x^2-1} = 2x-5+\frac{x}{x^2-1}.$$

So

$$\int \frac{2x^3-5x^2-x+5}{x^2-1}\,dx = \int \left(2x-5+\frac{x}{x^2-1}\right)\,dx.$$

Integrate $\dfrac{x}{x^2-1}$ by substitution:

$$\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln|x^2-1|+C.$$
$$[u = x^2-1, \quad du = 2x\,dx]$$

Hence,

$$\int \left(2x-5+\frac{x}{x^2-1}\right)\,dx = x^2-5x+\frac{1}{2}\ln|x^2-1|+C.\ \square$$

2. $\displaystyle\int \frac{x^2}{x^2-1}\,dx$

The top and the bottom have the same degree. Divide the bottom into the top:

$$\frac{x^2}{x^2-1} = 1 + \frac{1}{x^2-1}.$$

Therefore,

$$\int \frac{x^2}{x^2-1}\,dx = \int \left(1 + \frac{1}{x^2-1}\right)\,dx.$$

We are finished yet: The integral is passed along to the next step in the partial fractions procedure. □

3. $\displaystyle\int \frac{7x^2-25x+20}{x(x-2)^2}\,dx$

In this case, the top has degree 2 while the bottom has degree 3. We do not need to divide. The integral is passed along to the next step in the partial fractions procedure. □

Step 1 is a preliminary operation which puts the integral into a good form for the rest of the procedure. Before going on, check to see whether you can use a simple technique (like substitution) to do the integrals you've obtained. For example, in the first example above, we were able to complete the integration immediately. If you don't see a simple way to finish, then proceed to Step 2.

Step 2. You will now have an integral that looks like

$$\int \frac{P(x)}{Q(x)}\,dx,$$

where $P(x)$ and $Q(x)$ are polynomials, and the top is smaller in degree than the bottom. (The division process may spew out some other junk, but that can be integrated immediately. You only need to worry about the remaining fractional part.)

Factor the bottom of the fraction into a product of **linear** terms and **irreducible quadratic** terms.

Let's pause to explain our terminology. A **linear** term is a term where the variable occurs only to the first power. Here are some linear terms:

$$x-1, \quad 2x-3, \quad x.$$

An **irreducible quadratic** term is a quadratic term with only imaginary (complex) roots. That is, it is a quadratic which "doesn't factor". Here are some irreducible quadratic terms:

$$x^2+1, \quad x^2-2x+5.$$

You can check that a quadratic is irreducible by using the general quadratic formula to find its roots.

Warning: x^2-2 is *not* irreducible:

$$x^2-2 = (x-\sqrt{2})(x+\sqrt{2}).$$

("Ugly" factors are allowed.) And x^2 is not considered an irreducible quadratic: It is $(x-0)(x-0)$, the square of a linear term. This distinction will become important in Step 3. □

Examples.

1. In the second example, above, we obtained

$$\int \left(1 + \frac{1}{x^2-1}\right)\,dx.$$

We don't need to worry about the "1", since it integrates immediately to x. So consider $\int \frac{1}{x^2-1}\,dx$. Factor the denominator of the fraction:

$$\int \frac{1}{x^2-1}\,dx = \int \frac{1}{(x-1)(x+1)}\,dx.$$

We pass the integral along to the next step. □

2. In a previous example, we obtained

$$\int \frac{7x^2-25x+20}{x(x-2)^2}\,dx.$$

The bottom is already factored. Note that $(x-2)^2$ is *not* considered a quadratic factor; it's considered to be a linear factor raised to a power. □

3. $\int \frac{1}{x^3-1}\,dx$

Factor the denominator of the fraction:

$$\int \frac{1}{x^3-1}\,dx = \int \frac{1}{(x-1)(x^2+x+1)}\,dx.$$

Note that x^2+x+1 has only imaginary roots. □

Remark. It is known that any polynomial can be factored into linear terms and irreducible quadratic terms. However, the "can be" in the last sentence does not mean that you can always do it in practice! In fact, another branch of mathematics called **Galois theory** tells us that there is no general formula for finding exactly the roots of polynomials of degree 5 or higher.

We are interested in understanding *how partial fractions works*; for that reason, the polynomials in our problems will usually be easy to factor. Moreover, programs like *Mathematica* can approximate roots, and that is often good enough for applications. However, you should keep in mind that factoring polynomials is, in general, not a simple thing. □

Step 3. Obtain the **partial fractions decomposition** for the fraction.

This is the heart of the partial fractions method. It is basically a lot of algebra, but it's sufficiently complicated that the best way to describe it is by doing some examples.

Examples.

1. $\int \frac{1}{(x-1)(x+1)}\,dx$

We want to find numbers A and B such that

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

is an **algebraic identity**. This means that the equation should be true for all values of x for which both sides are defined.

First, multiply through to clear denominators.

$$1 = A(x+1) + B(x-1).$$

Since this equation is to be true for *all* x, it must true for $x = 1$. Plug in $x = 1$. We obtain $1 = 2A$, so $A = 1/2$.

I chose $x = 1$ because it *killed* the $B(x-1)$-term, and this allowed me to solve for A.

I see that setting $x = -1$ will kill the $A(x+1)$ term. Doing so, I get $1 = 2B$, so $B = 1/2$. Therefore,

$$\frac{1}{(x-1)(x+1)} = \frac{1}{2}\,\frac{1}{x-1} + \frac{1}{2}\,\frac{1}{x+1}.$$

(Check: If you add the fractions on the right, you'll get the fraction on the left.)
Now we can do the integral:

$$\int \frac{1}{(x-1)(x+1)}\,dx = \int \left(\frac{1}{2}\,\frac{1}{x-1} + \frac{1}{2}\,\frac{1}{x+1}\right) = \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C. \square$$

Remark. The following integral often comes up in these problems:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C, \quad a \neq 0.$$

(Do you see how to work it out? Substitute $u = ax + b$, so $du = a\,dx$.) For example,

$$\int \frac{1}{7x+5}\,dx = \frac{1}{7}\ln|7x+5| + C,$$
$$\int \frac{1}{3-2x}\,dx = -\frac{1}{2}\ln|3-2x| + C. \square$$

2. $\displaystyle\int \frac{7x^2 - 25x + 20}{x(x-2)^2}\,dx$

This example will show how we handle **repeated factors**. In this case, the **repeated factor** is the "$(x-2)^2$". We try to find A, B, and C so that

$$\frac{7x^2 - 25x + 20}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}.$$

For a **repeated factor**, you have one term for each power up to the power the factor is raised to.

(Why do you need both an "$x-2$" and an "$(x-2)^2$"? Think about it. We are trying to find fractions which add up to $\dfrac{7x^2 - 25x + 20}{x(x-2)^2}$. Those fractions could have an $x - 2$ or an $(x-2)^2$ on the bottom, because both lead to an $(x-2)^2$ in the "common denominator". We have to allow for all the possibilities.)

From here, we proceed as we did in the first example. Clear denominators:

$$7x^2 - 25x + 20 = A(x-2)^2 + Bx(x-2) + Cx.$$

We see that plugging in $x = 0$ and $x = 2$ will kill lots of terms. First, set $x = 0$. We obtain $20 = 4A + 0 + 0$, so $A = 5$.

Next, set $x = 2$. We get $28 - 50 + 20 = 0 + 0 + 2C$ so $C = -1$.

Now that we've run out of "good" numbers to plug in, there are several ways to proceed. First, put $A = 5$ and $C = -1$ into the equation:

$$7x^2 - 25x + 20 = 5(x-2)^2 + Bx(x-2) - x.$$

One approach is to pick a number at random (other than $x = 0$ or $x = 2$) and plug it in for x. Since only B is left, we can solve the resulting equation for B. For example, set $x = 1$. We get $2 = 5 - B - 1$, so $B = 2$.

Another approach uses a little trick. Since the equation is supposed to be an identity, we can *differentiate* both sides:

$$14x - 25 = 10(x-2) + Bx + B(x-2) - 1.$$

The new equation is *simpler*, because differentiation reduces powers. Now plug in something for x — for example, $x = 0$ gives $-25 = -20 + 0 - 2B - 1$, so $B = 2$.

You can even differentiate again! Differentiating $14x - 25 = 10(x-2) + Bx + B(x-2) - 1$ gives $14 = 10 + B + B$, and again $B = 2$.

You can plug in numbers or differentiate as you wish. The idea is to avoid doing complicated algebra — such as multiplying stuff out — because you are likely to make mistakes.

Anyway, we can substitute our A, B, and C values into the original decomposition:

$$\frac{7x^2 - 25x + 20}{x(x-2)^2} = \frac{5}{x} + \frac{2}{x-2} - \frac{1}{(x-2)^2}.$$

Now we can do the integral:

$$\int \frac{7x^2 - 25x + 20}{x(x-2)^2}\, dx = \int \left(\frac{5}{x} + \frac{2}{x-2} - \frac{1}{(x-2)^2}\right) dx = 5 \ln |x| + 2 \ln |x-2| + \frac{1}{x-2} + C. \square$$

3. How would you try to decompose

$$\frac{3x^3 + 4x - 17}{x^3(2x-1)^2}$$

using partial fractions? That is, what is the initial partial fractions equation?

The linear factor x is repeated 3 times and the linear factor $2x - 1$ is repeated twice. We should try to solve

$$\frac{3x^3 + 4x - 17}{x^3(2x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{2x-1} + \frac{E}{(2x-1)^2}.$$

Notice that *the top of the fraction is irrelevant in deciding how to set up the decomposition.* It only comes in during the solution process.

Notice also that "x^3" is considered a linear term (x) raised to the third power. We get one term on the right for x, one for x^2, and one for x^3 — no "skipping"! $\square$

4. $\displaystyle\int \frac{1}{(x-1)(x^2+x+1)}\, dx$

In this example, we have an irreducible quadratic factor $x^2 + x + 1$. Here is what we do with it:

$$\frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + x + 1}.$$

Thus, a quadratic factor (or a quadratic factor to a power) will produce terms on the right with "two letters" on top.

The rationale is the same as the one we gave for repeated factors. We don't know what kind of fraction to expect, so we have to take the most general case.

(You might ask: "Well, why not try $\dfrac{Bx^2 + Cx + D}{x^2 + x + 1}$ — or higher powers?" The answer is that if we had a quadratic or something bigger on top, we could divide first to reduce to a $Bx + C$ form.)

The solution procedure is similar to those we used above. Clear denominators:

$$1 = A(x^2 + x + 1) + (Bx + C)(x - 1).$$

Set $x = 1$. We get $1 = 3A$, so $A = 1/3$. Plug it back in:

$$1 = \frac{1}{3}(x^2 + x + 1) + (Bx + C)(x - 1).$$

$x = 0$ will kill B, leaving C to be solved for. Setting $x = 0$, we get $1 = 1/3 - C$, so $C = -2/3$. Plug it back in:

$$1 = \frac{1}{3}(x^2 + x + 1) + (Bx - \frac{2}{3})(x - 1).$$

Now we can either plug in a value for x at random, or differentiate. Let's differentiate:

$$0 = \frac{1}{3}(2x+1) + (Bx - \frac{2}{3}) + B(x-1).$$

Differentiate again:

$$0 = \frac{2}{3} + B + B.$$

We get $B = -1/3$.
We take our winnings and plug them back into the original fractional decomposition.

$$\frac{1}{(x-1)(x^2+x+1)} = \frac{1}{3}\,\frac{1}{x-1} - \frac{1}{3}\,\frac{x+2}{x^2+x+1}.$$

Our integral is

$$\int \frac{1}{(x-1)(x^2+x+1)}\,dx = \int \left(\frac{1}{3}\,\frac{1}{x-1} - \frac{1}{3}\,\frac{x+2}{x^2+x+1}\right)\,dx.$$

We'll do the integrals separately. First,

$$\int \frac{1}{3}\,\frac{1}{x-1} = \frac{1}{3}\ln|x-1| + C.$$

Next,

$$\frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx = \frac{1}{6}\int \frac{2x+4}{x^2+x+1}\,dx =$$
$$\frac{1}{6}\int \frac{2x+1+3}{x^2+x+1}\,dx = \frac{1}{6}\left(\int \frac{2x+1}{x^2+x+1}\,dx + 3\int \frac{1}{x^2+x+1}\,dx\right).$$

The first integral succumbs to a substitution:

$$\int \frac{2x+1}{x^2+x+1}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|x^2+x+1| + C.$$
$$[u = x^2+x+1, \quad du = (2x+1)\,dx]$$

The second requires completing the square:

$$\int \frac{1}{x^2+x+1}\,dx = \int \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}}\,dx = \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx = \frac{\sqrt{3}}{2}\int \frac{1}{\frac{3}{4}u^2+\frac{3}{4}}\,du =$$
$$[x+\frac{1}{2} = \frac{\sqrt{3}}{2}u, \quad dx = \frac{\sqrt{3}}{2}\,du]$$
$$\frac{2}{\sqrt{3}}\int \frac{1}{u^2+1}\,du = \frac{2}{\sqrt{3}}\arctan u + C = \frac{2}{\sqrt{3}}\arctan\frac{2(x+\frac{1}{2})}{\sqrt{3}} + C.$$

(Whew!) Putting the two together, we get

$$\frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx = \frac{1}{6}\ln|x^2+x+1| + \frac{1}{\sqrt{3}}\arctan\frac{2(x+\frac{1}{2})}{\sqrt{3}} + C.$$

Finally, we get the answer to the original problem:

$$\int \frac{1}{(x-1)(x^2+x+1)}\,dx = \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\arctan\frac{2(x+\frac{1}{2})}{\sqrt{3}} + C.$$

What a mess! This is why you should consider other methods before you turn to partial fractions! $\square$

Problems on Partial Fractions.

1. How would you try to decompose

$$\frac{2(x-2)^2}{x^3(x^2+4)^2}$$

using partial fractions?

2. Why do you try the partial fractions decomposition

$$\frac{5x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

rather than the decomposition

$$\frac{5x}{(x-1)^2(x+1)} = \frac{A}{(x-1)^2} + \frac{B}{x+1}?$$

3. What is wrong with the following "partial fractions decomposition"?

$$\frac{7}{x(x-1)} = \frac{A}{x} + \frac{B}{x} + \frac{C}{x-1}$$

4. Find the partial fractions decomposition of

$$\frac{-3x^4+x^3-6x^2-3}{x(x^2+1)^2}.$$

5. (a) Compute $\displaystyle\int \frac{-x^2+8x-4}{x(x^2+x-2)}\,dx$.

(b) If we use the antiderivative we found in (a) to compute

$$\int_{-1}^{\frac{1}{2}} \frac{-x^2+8x-4}{x(x^2+x-2)}\,dx,$$

we obtain

$$[2\ln|x| + \ln|x-1| - 4\ln|x+2|]_{-1}^{\frac{1}{2}} = \left(2\ln\frac{1}{2} + \ln\frac{1}{2} - 4\ln 32\right) - (2\ln 1 + \ln 2 - 4\ln 1) \approx -4.39445.$$

Does this computation make sense? Why or why not?

6. Compute $\displaystyle\int \frac{3+4x+5x^2+3x^3}{x^2(x+3)}\,dx$.

Solutions to the Problems on Partial Fractions.

1. How would you try to decompose

$$\frac{2(x-2)^2}{x^3(x^2+4)^2}$$

using partial fractions?

$$\frac{2(x-2)^2}{x^3(x^2+4)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+4} + \frac{Fx+G}{(x^2+4)^2}. \square$$

2. Why do you try the partial fractions decomposition

$$\frac{5x}{(x-1)^2(x+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1}$$

rather than the decomposition

$$\frac{5x}{(x-1)^2(x+1)} = \frac{A}{(x-1)^2} + \frac{B}{x+1}?$$

Partial fractions is the opposite of combining fractions over a common denominator. Hence, we ask "What fractions would add up to $\frac{5x}{(x-1)^2(x+1)}$?" Both of the decompositions above could occur, since both have $(x-1)^2(x+1)$ as the common denominator. Since we don't know beforehand what the fractions are, we must assume the 'worst case'. □

3. What is wrong with the following "partial fractions decomposition"?

$$\frac{7}{x(x-1)} = \frac{A}{x} + \frac{B}{x} + \frac{C}{x-1}$$

The first two terms could be combined into a single term $\frac{D}{x}$ and are hence redundant. There is no reason to list a given denominator twice. □

4. Find the partial fractions decomposition of

$$\frac{-3x^4 + x^3 - 6x^2 - 3}{x(x^2+1)^2}.$$

We have a linear factor x and a repeated quadratic factor $(x^2+1)^2$:

$$\frac{-3x^4 + x^3 - 6x^2 - 3}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}.$$

Clear denominators:

$$-3x^4 + x^3 - 6x^2 - 3 = A(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)(x).$$

Set $x = 0$. We get $A = -3$. Plug it back in:

$$-3x^4 + x^3 - 6x^2 - 3 = -3(x^2+1)^2 + (Bx+C)(x)(x^2+1) + (Dx+E)(x).$$

Differentiate:

$$12x^3 + 3x^2 - 12x = -12x(x^2+1) + B(x^3+x) + (Bx+C)(3x^2+1) + 2Dx + E.$$

Set $x = 0$. We get $C + E = 0$.
Differentiate again:

$$-36x^2 + 6x - 12 = -36x^2 - 12 + B(3x^2+1) + B(3x^2+1) + (Bx+C)(6x) + 2D.$$

Set $x = 0$. We get $B + D = 0$.
Cancel the $-36x^2$ and -12 terms in the previous equation, then differentiate:

$$\begin{aligned} 6x &= B(3x^2+1) + B(3x^2+1) + (Bx+C)(6x) + 2D, \\ 6 &= 6Bx + 6Bx + 6Bx + (Bx+C)(6). \end{aligned}$$

Set $x = 0$. We get $C = 1$. Since $C + E = 0$, we have $E = -1$.

Plug $C = 1$ back in, then simplify the equation. We get $6 = 24Bx + 6$, or $0 = 24Bx$. Set $x = 1$; we get $B = 0$. But $B + D = 0$, so $D = 0$.

Hence, our decomposition is

$$\frac{-3x^4 + x^3 - 6x^2 - 3}{x(x^2+1)^2} = -\frac{3}{x} + \frac{1}{x^2+1} - \frac{1}{(x^2+1)^2}. \square$$

5. (a) Compute $\displaystyle\int \frac{-x^2 + 8x - 4}{x(x^2 + x - 2)}\, dx$.

$x(x^2 + x - 2) = x(x-1)(x+2)$, so we try

$$\frac{-x^2 + 8x - 4}{x(x^2 + x - 2)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}.$$

Clear denominators:

$$-x^2 + 8x - 4 = A(x-1)(x+2) + Bx(x+2) + Cx(x-1).$$

Set $x = 0$. We get $-4 = -2A$, or $A = 2$.
Set $x = 1$. We get $3 = 3B$, or $B = 1$.
Set $x = -2$. We get $-24 = 6C$, or $C = -4$.
Therefore,

$$\int \frac{-x^2 + 8x - 4}{x(x^2 + x - 2)}\, dx = \int \left(\frac{2}{x} + \frac{1}{x-1} - \frac{4}{x+2}\right) dx = 2\ln|x| + \ln|x-1| - 4\ln|x+2| + C. \square$$

(b) If we use the antiderivative we found in (a) to compute

$$\int_{-1}^{\frac{1}{2}} \frac{-x^2 + 8x - 4}{x(x^2 + x - 2)}\, dx,$$

we obtain

$$[2\ln|x| + \ln|x-1| - 4\ln|x+2|]_{-1}^{\frac{1}{2}} = \left(2\ln\frac{1}{2} + \ln\frac{1}{2} - 4\ln 32\right) - (2\ln 1 + \ln 2 - 4\ln 1) \approx -4.39445.$$

Does this computation make sense? Why or why not?

The computation is incorrect, because our antiderivative is valid only within intervals which avoid the singularities at $x = -2$, $x = 0$, and $x = 1$. The range of the integral above includes the singularity at $x = 0$. It is not legal to simply "plug in the endpoints". $\square$

6. Compute $\displaystyle\int \frac{3 + 4x + 5x^2 + 3x^3}{x^2(x+3)}\, dx$.

We divide first:

$$\frac{3 + 4x + 5x^2 + 3x^3}{x^2(x+3)} = 3 + \frac{3 + 4x - 4x^2}{x^2(x+3)}.$$

We try

$$\frac{3 + 4x - 4x^2}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}.$$

Clear denominators:

$$3 + 4x - 4x^2 = Ax(x+3) + B(x+3) + Cx^2.$$

Set $x = 0$. We get $3 = 3B$, so $B = 1$. Set $x = -3$. We get $-45 = 9C$, so $C = -5$. Plug these back in:

$$3 + 4x - 4x^2 = Ax(x+3) + (x+3) - 5x^2.$$

Differentiate:

$$4 - 8x = A(x+3) + Ax + 1 - 10x.$$

Set $x = 0$. We get $4 = 3A + 1$, so $A = 1$.

$$\frac{3 + 4x - 4x^2}{x^2(x+3)} = \frac{1}{x} + \frac{1}{x^2} - \frac{5}{x+3},$$

so

$$\frac{3 + 4x + 5x^2 + 3x^3}{x^2(x+3)} = 3 + \frac{1}{x} + \frac{1}{x^2} - \frac{5}{x+3}.$$

Therefore,

$$\int \frac{3 + 4x + 5x^2 + 3x^3}{x^2(x+3)}\,dx = \int \left(3 + \frac{1}{x} + \frac{1}{x^2} - \frac{5}{x+3}\right) dx = 3x + \ln|x| - \frac{1}{x} - 5\ln|x+3| + C. \square$$