10-12-1993

Lines in space.

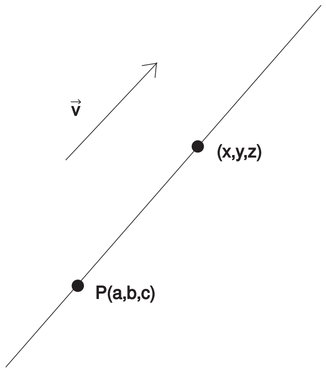
We're all familiar with the idea that "two points determine a line". In three dimensions, it's better to think of a line as determined by a point and a vector.

> A line in three dimensions is uniquely determined by **a point on the line** and **a vector parallel to the line**.

To understand why this works, imagine a vector — an arrow — in three-dimensional space. The lines parallel to the vector are "lined up" like the noodles in a box of spaghetti.

Thus, there are infinitely many lines ("noodles") parallel to the given vector. However, only one of those lines passes through the given point. (Imagine pulling a noodle out of the box till it hits the point.)

Thus, a point on the line and a vector parallel to the line *really do* determine the line. Next, we'd like to obtain the equation(s) of the line from the point and the vector. Suppose the point is $P(a,b,c)$ and the vector is $\vec{v} = \langle v_1, v_2, v_3 \rangle$. What condition must be satisfied by a point (x,y,z) on the line?



As the picture shows, the vector from P to (x,y,z) must be parallel to $\vec{v}$. The vector from P to (x,y,z) is $\langle x-a, y-b, z-c \rangle$, and parallel vectors are multiples. Therefore, we have

$$\langle x-a, y-b, z-c \rangle = t\langle v_1, v_2, v_3 \rangle.$$

This is called the **vector equation of the line**. If we multiply out and equate components, we obtain

$$x - a = v_1 t, \quad y - b = v_2 t, \quad z - c = v_3 t,$$

the **parametric equations for the line**.

Notice that a line is a special kind of parametrized curve. In fact, the vector $\vec{v}$ is nothing but the *velocity vector* for the line, considered as a curve.

In addition, notice the resemblance between the parametric equations and the equation for a line in the plane in point-slope form $y - y_0 = m(x - x_0)$. In each case, the point on the line is *subtracted* from the variables. The slope, which corresponds to the parallel vector, *multiplies* the difference.

Examples.

1. Find three points on the line

$$x = t, \quad y = 2 + 3t, \quad z = 4 - 2t.$$

Find a vector parallel to the line.

We obtain points on the line by plugging in values of t. To pick values at random,

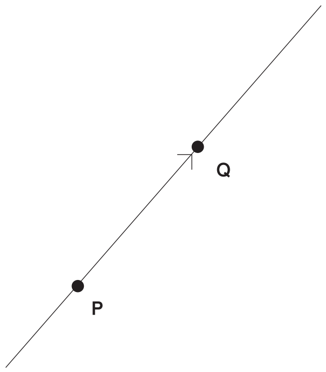
$$\begin{array}{rlll} t = 0: & x = 0, & y = 2, & z = 4, \\ t = 1: & x = 1, & y = 5, & z = 2, \\ t = -2: & x = -2, & y = -4, & z = 8. \end{array}$$

Hence, the points $(0, 2, 4)$, $(1, 5, 2)$, and $(-2, -4, 8)$ are on the line.

We can read off a vector parallel to the line by reading off the coefficients of t: $\langle 1, 3, -2 \rangle$ is parallel to the line. □

2. Find the equation of the line which passes through the points $P(3, 4, 6)$ and $Q(-1, 3, 2)$.

We need a point on the line and a vector parallel to the line. We can use either P or Q as our point; let's use P. Since P and Q are on the line, the vector $\vec{PQ}$ is parallel to the line.



Now $\vec{PQ} = \langle -4, -1, -4 \rangle$, so the line is

$$x - 3 = -4t, \quad y - 4 = -t, \quad z - 6 = -4t. \ \square$$

3. Determine whether the following lines intersect. If they do, find the point of intersection.

$$\begin{array}{lll} x = 2 + t, & y = 3 - t, & z = 4 + 2t \\ x = 1 + s, & y = 6 - 2s, & z = 3s \end{array}$$

To determine where two objects intersect, we usually solve their equations simultaneously. That's what we do here, but there is a little trick that you must watch out for.

First, equate the two expressions for x: $2 + t = x = 1 + s$, so $1 + t = s$.

Equate the two expressions for y, then substitute $s = 1 + t$: $3 - t = y = 6 - 2s$, so $3 - t = 6 - 2(1 + t)$ and $t = 1$. $t = 1$ gives $s = 1 + t = 2$.

Here is where it's easy to make a mistake. With t and s known, it's tempting to plug t (or s) back into the x-y-z equations, find x, y, and z, and conclude that the intersection point has been found.

A little thought shows that this can't be the whole story. Suppose the two lines *don't* intersect. Where in the above procedure would we find this out? There must be something more.

We notice that we never made use of the z equations. That's the key: The values for t and s must be substituted into the z equations and checked for consistency.

Since $t = 1$ and $s = 2$, we have $z = 4 + 2t = 6$ and $z = 3s = 6$.

Since the z-values check, the lines *do* intersect, and we can find the intersection point by plugging t (or s) back into the x-y-z equations:

$$x = 2 + 1 = 3, \quad y = 3 - 1 = 2, \quad z = 4 + 2 = 6.$$

Therefore, the lines intersect at the point $(3, 2, 6)$.

If you reach a contradiction at any point in the process of finding t and s — something like "$0 = 1$", for instance — then the lines *don't* intersect. $\Box$

4. (a) Show that the lines

$$x = 1 - t, \quad y = 2 + t, \quad z = 2 + 2t$$
$$x = 1 + 2s, \quad y = 3 - 2s, \quad z = 5 - 4s$$

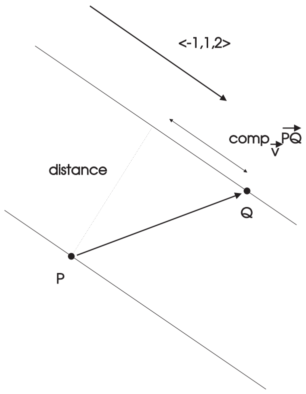
are parallel.

When you write down the equation of a line, the components of the line's parallel vector multiply the line parameter. You can also go backwards: If you have the parametric equations of a line, the coefficients of the parameter make up the components of a vector parallel to the line.

Thus, $\langle -1, 1, 2\rangle$ is parallel to the first line and $\langle 2, -2, -4\rangle$ is parallel to the second line.

Two vectors are parallel if and only if they're multiples of one another. Since $\langle 2, -2, -4\rangle = (-2) \cdot \langle -1, 1, 2\rangle$, the two line vectors are parallel. These vectors are parallel to their respective lines, so the lines are parallel as well. $\Box$

(b) Find the distance between the lines in part (a).



As the picture shows, we think of the distance as one leg of a right triangle. We will find it using Pythagoras.

First, the vector $\langle -1, 1, 2\rangle$ is parallel to the lines.

Set $t = 0$. The point $P(1, 2, 2)$ is on the first line. Set $s = 0$. The point $Q(1, 3, 5)$ is on the second line. (You can use any values for t and s, but 0 makes the computations easy.) Therefore, $\overrightarrow{PQ} = \langle 0, 1, 3\rangle$.

$\overrightarrow{PQ}$ gives us the hypotenuse of the right triangle. We get one of the legs by taking the component of $\overrightarrow{PQ}$ in the direction of $\vec{v} = \langle -1, 1, 2\rangle$. (This is, roughly, the "shadow" that $\overrightarrow{PQ}$ makes on $\vec{v}$.)

$$\operatorname{comp}_{\vec{v}} \overrightarrow{PQ} = \frac{\overrightarrow{PQ} \cdot \vec{v}}{|\vec{v}|} = \frac{\langle 0, 1, 3\rangle \cdot \langle -1, 1, 2\rangle}{|\langle -1, 1, 2\rangle|} = \frac{7}{\sqrt{6}}.$$

Now find the distance using Pythagoras:

$$\text{distance} = \sqrt{|\overrightarrow{PQ}|^2 - (\operatorname{comp}_{\vec{v}} \overrightarrow{PQ})^2} = \sqrt{10 - \frac{49}{6}} = \sqrt{\frac{11}{6}} \approx 1.35401. \Box$$

Problems on Lines in Space.

1. Find the distance from the point $P(1, 0, -3)$ to the line

$$x = 1 + 2t, \quad y = -t, \quad z = 2 + 2t.$$

First, find a point Q on the line. Find the vector $\vec{PQ}$. Find a vector $\vec{v}$ parallel to the line. Find $\text{comp}_{\vec{v}}\, \vec{PQ}$, the scalar component of $\vec{PQ}$ in the direction of $\vec{v}$. Finally, use Pythagoras to find the distance. Draw a picture to see what's going on!

2. (a) Find the equation of the line which passes through the points $(2, -1, 3)$ and $(4, 2, 1)$.

(b) Where does the line in part (a) intersect the y-z plane? (Hint: The y-z plane is $x = 0$.)

3. (a) Find the equation of the line which goes through the point $(5, 1, 4)$ and is parallel to the vector $\langle 1, 2, -2\rangle$.

(b) Find the point on the line in part (a) which is closest to the origin.

Use calculus! The square of the distance from the origin to (x, y, z) is

$$S = (x - 0)^2 + (y - 0)^2 + (z - 0)^2 = x^2 + y^2 + z^2 .$$

Plug the line equations into the equation for S to obtain an equation for S in terms of t. Now drag out your max-min apparatus to find the value of t for which S is a minimum.

4. (a) Show that the lines

$$\begin{aligned} x = 1 + t, \quad & y = 2 + 2t, \quad z = 4t \\ x = 2s, \quad & y = 1 - s, \quad z = 3s \end{aligned}$$

are skew.

(b) Find two unit vectors which are perpendicular to both lines.

5. Find the line which passes through the point $(2, -3, 1)$ and is parallel to the line containing the points $(3, 9, 5)$ and $(4, 7, 2)$.

6. In each case, determine whether the pairs of lines intersect. If they do, find the point of intersection.

(a)

$$\begin{aligned} x = 1 + 2t, \quad & y = 1 - 4t, \quad z = 5 - t \\ x = 4 - v, \quad & y = -1 + 6v, \quad z = 4 + v \end{aligned}$$

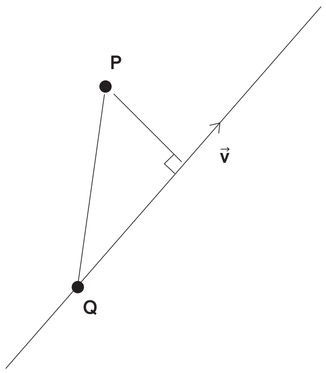
(b)

$$\begin{aligned} x = 3 + t, \quad & y = 2 - 4t, \quad z = t \\ x = 4 - s, \quad & y = 3 + s, \quad z = -2 + 3s \end{aligned}$$

Solutions to the Problems on Lines in Space.

1. Find the distance from the point $P(1,0,-3)$ to the line

$$x = 1 + 2t, \quad y = -t, \quad z = 2 + 2t.$$



Let $t = 0$. We find that $Q(1,0,2)$ is a point on the line. Therefore, $\vec{PQ} = \langle 0,0,5\rangle$. Reading off the coefficients of t, we see that $\vec{v} = \langle 2,-1,2\rangle$ is parallel to the line. Therefore,

$$\operatorname{comp}_{\vec{v}} \vec{PQ} = \frac{\langle 0,0,5\rangle \cdot \langle 2,-1,2\rangle}{|\langle 2,-1,2\rangle|} = \frac{10}{3}.$$

The distance from P to the line is the leg of a right triangle, and we know the hypotenuse ($|\vec{PQ}|$) and the other leg ($\operatorname{comp}_{\vec{v}} \vec{PQ}$). Hence, we may find the distance using Pythagoras:

$$\text{distance} = \left\{|\vec{PQ}|^2 - \left(\operatorname{comp}_{\vec{v}} \vec{PQ}\right)^2\right\}^{1/2} = \left(25 - \frac{100}{9}\right)^{1/2} = \frac{5\sqrt{5}}{3} \approx 3.72678. \square$$

2. (a) Find the equation of the line which passes through the points $(2,-1,3)$ and $(4,2,1)$.

We need a point on the line. We'll use $(2,-1,3)$.

We need a vector parallel to the line. The vector from the first point to the second is $\langle 2,3,-2\rangle$; this is surely parallel to the line.

The line is

$$x - 2 = 2t, \quad y + 1 = 3t, \quad z - 3 = -2t. \square$$

(b) Where does the line in part (a) intersect the y-z plane?

The y-z-plane is $x = 0$, so set $x = 0$. The first equation gives $-2 = 2t$, or $t = -1$. Plug this into the y and z equations: We get $y = -4$, $z = 5$. Therefore, the line intersects the y-z plane at $(0,-4,5)$. $\square$

3. (a) Find the equation of the line which goes through the point $(5,1,4)$ and is parallel to the vector $\langle 1,2,-2\rangle$.

$$x - 5 = t, \quad y - 1 = 2t, \quad z - 4 = -2t. \square$$

(b) Find the point on the line in part (a) which is closest to the origin.

Following the hint, we compute the (square of the) distance from a point on the line to the origin:

$$S = x^2 + y^2 + z^2 = (5+t)^2 + (1+2t)^2 + (4-2t)^2,$$

$$\frac{dS}{dt} = 2(5+t) + 4(1+2t) - 4(4-2t) = -2 + 18t.$$

Set $\dfrac{dS}{dt} = 0$. We obtain $-2 + 18t = 0$, or $t = \dfrac{1}{9}$. Since the equation of S is quadratic in t, and since the t^2 term has a positive coefficient, the critical point is an absolute min.

We find the closest point on the line by plugging $t = \dfrac{1}{9}$ into the x-y-z equations: $x = \dfrac{46}{9}$, $y = \dfrac{11}{9}$, $z = \dfrac{34}{9}$. The closest point is $\left(\dfrac{46}{9}, \dfrac{11}{9}, \dfrac{34}{9}\right)$. $\square$

4. (a) Show that the lines

$$x = 1 + t, \quad y = 2 + 2t, \quad z = 4t$$
$$x = 2s, \quad y = 1 - s, \quad z = 3s$$

are skew.

Two lines are *skew* if they're not parallel and they don't intersect.

The first line is parallel to $\langle 1, 2, 4\rangle$. The second line is parallel to $\langle 2, -1, 3$. The vectors aren't multiples, so the vectors aren't parallel. Hence, the lines aren't parallel.

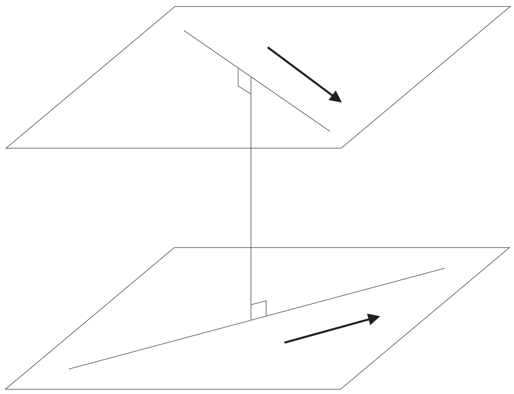
Next, I'll show that the lines don't intersect. Set the x-expressions equal: $1 + t = x = 2s$, so $t = 2s - 1$.

Equate the y-expressions and substitute $t = 2s - 1$: $2 + 2t = y = 1 - s$, so $2 + 2(2s - 1) = 1 - s$, and $s = \dfrac{1}{5}$. Therefore, $t = -\dfrac{3}{5}$.

Plug s and t into the z equations. We get $z = 4t = -\dfrac{12}{5}$ but $z = 3s = \dfrac{3}{5}$. The z values are inconsistent, so the lines don't intersect.

Therefore, the lines are skew. $\square$

(b) Find two unit vectors which are perpendicular to both lines.



The vector $\langle 1, 2, 4\rangle$ is parallel to the first line. The vector $\langle 2, -1, 3\rangle$ is parallel to the second line. The cross product of the vectors is perpendicular to the two vectors, hence perpendicular to the two lines:

$$\langle 1, 2, 4\rangle \times \langle 2, -1, 3\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 4 \\ 2 & -1 & 3 \end{vmatrix} = \langle 10, 5, -5\rangle.$$

$|\langle 10, 5, -5\rangle| = 5\sqrt{6}$, so $\dfrac{1}{\sqrt{6}}\langle 2, 1, -1\rangle$ and $-\dfrac{1}{\sqrt{6}}\langle 2, 1, -1\rangle$ are unit vectors perpendicular to the lines. $\square$

5. Find the line which passes through the point $(2, -3, 1)$ and is parallel to the line containing the points $(3, 9, 5)$ and $(4, 7, 2)$.

Since the lines are parallel, they have the same parallel vector. The vector $\langle 1, -2, -3\rangle$ is parallel to the second line, since it's the vector from $(3, 9, 5)$ to $(4, 7, 2)$. Using this vector for the first line together with the point $(2, -3, 1)$, we obtain

$$x - 2 = t, \quad y + 3 = -2t, \quad z - 1 = -3t. \ \square$$

6. In each case, determine whether the pairs of lines intersect. If they do, find the point of intersection.

(a)
$$x = 1 + 2t, \quad y = 1 - 4t, \quad z = 5 - t$$
$$x = 4 - v, \quad y = -1 + 6v, \quad z = 4 + v$$

Equate the expressions for x: $1 + 2t = x = 4 - v$, so $v = 3 - 2t$.

Next, we equate the expressions for y, and substitute $v = 3 - 2t$: $1 - 4t = y = -1 + 6v$, so $1 - 4t = -1 + 6(3 - 2t)$, and $t = 2$. This gives $v = 3 - 2t = -1$.

Check the values by plugging them into the z equations: $z = 5 - t = 3$, and $z = 4 + v = 3$.

The equations are consistent, so the lines intersect. If we plug $t = 2$ into the x-y-z equations, we obtain $x = 5$, $y = -7$, $z = 3$. The lines intersect at $(5, -7, 3)$. $\square$

(b)
$$x = 3 + t, \quad y = 2 - 4t, \quad z = t$$
$$x = 4 - s, \quad y = 3 + s, \quad z = -2 + 3s$$

Equate the expressions for x: $3 + t = x = 4 - s$, so $t = 1 - s$.

Equate the expressions for y and substitute $t = 1 - s$: $2 - 4t = y = 3 + s$, so $2 - 4(1 - s) = 3 + s$ and $s = \frac{5}{3}$. This gives $t = 1 - s = -\frac{2}{3}$.

Finally, we plug s and t into the z equations: $z = t = -\frac{2}{3}$, but $z = -2 + 3s = 3$.

The equations are inconsistent, so the lines do not intersect. $\square$