3-10-1994

Laplace transforms.

The **Laplace transform** of $f(t)$ is

$$\mathcal{L}\left(f(t)\right)(s) = \int_0^{+\infty} e^{-st} f(t)\, dt.$$

Example. The Laplace transform of $f(t) = e^{at}$ is

$$\mathcal{L}\left(f(t)\right) = \int_0^{+\infty} e^{-st} e^{at}\, dt = \int_0^{+\infty} e^{-(s-a)t}\, dt =$$

$$\lim_{p\to+\infty} \left[-\frac{1}{s-a} e^{-(s-a)t}\right]_0^p = -\frac{1}{s-a} \lim_{p\to+\infty} \left(e^{-(s-a)p} - 1\right) = \frac{1}{s-a}. \quad \square$$

Thus, the Laplace transform replaces one function with another. Laplace transforms are defined in terms of integrals. Since integrals are **linear**, so are Laplace transforms:

$$\mathcal{L}\left(f_1 + f_2\right) = \mathcal{L}(f_1) + \mathcal{L}(f_2),$$

$$\mathcal{L}\left(c \cdot f\right) = c\,\mathcal{L}(f).$$

Here $f(t)$, $f_1(t)$, and $f_2(t)$ are functions and c is a constant.

In words, you can take the transform of a sum of terms by taking the transform of each term, and you can "pull constants out" of transforms in the way that you "pull them out" of derivatives and integrals.

Laplace transforms are often used to solve differential equations. Here's the idea:

$$\begin{array}{ccc}
\left\{\begin{matrix}\text{differential}\\ \text{equation}\end{matrix}\right\} & \xrightarrow{\mathcal{L}} & \left\{\begin{matrix}\text{algebraic}\\ \text{equation}\end{matrix}\right\} \\
 & & \downarrow \\
\left\{\begin{matrix}\text{solution to}\\ \text{differential}\\ \text{equation}\end{matrix}\right\} & \xleftarrow{\mathcal{L}^{-1}} & \left\{\begin{matrix}\text{solution to}\\ \text{algebraic}\\ \text{equation}\end{matrix}\right\}
\end{array}$$

Starting with a differential equation (usually with initial conditions), take the Laplace transform $\mathcal{L}$, obtaining an algebraic equation. Algebraic equations are usually easier to solve than differential equations. Solve the algebraic equation. Finally, apply the **inverse transform** $\mathcal{L}^{-1}$ to the solution to the algebraic equation to obtain a solution to the differential equation.

The relationship between computing transforms and computing inverse transforms is like the relationship between differentiation and antidifferentiation. On the one hand, the derivative of $\sin x$ is $\cos x$ (differentiation). On the other hand, $\cos x$ is obtained by differentiating $\sin x$ (antidifferentiation).

Similarly, the transform of e^{2t} is $\dfrac{1}{s-2}$ (as you can verify by applying the formula from in the previous example), and the inverse transform of $\dfrac{1}{s-2}$ is e^{2t}.

I'll use a **table of transforms** in doing the problems. You can consider this table as *given*; that is, you are not expected to memorize these formulas. On the other hand, it would be fair to expect you to be able to *derive* one of the formulas from scratch.

Table of Laplace Transforms.

$f(t)$	$F(s) = \mathcal{L}\left(f(t)\right)$
1	$\dfrac{1}{s}, \qquad s > 0$
e^{at}	$\dfrac{1}{s-a}, \qquad s > a$
$t^n, \qquad n \in \mathbf{Z}^+$	$\dfrac{n!}{s^{n+1}}, \qquad s > 0$
$t^r, \qquad r > 0$	$\dfrac{\Gamma(r+1)}{s^{r+1}}, \qquad s > 0$
$\sin at$	$\dfrac{a}{s^2+a^2}, \qquad s > 0$
$\cos at$	$\dfrac{s}{s^2+a^2}, \qquad s > 0$
$\sinh at$	$\dfrac{a}{s^2-a^2}, \qquad s > \lvert a\rvert$
$\cosh at$	$\dfrac{s}{s^2-a^2}, \qquad s > \lvert a\rvert$
$e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \qquad s > a$
$e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \qquad s > a$
$t^n e^{at}, \qquad n \in \mathbf{Z}^+$	$\dfrac{n!}{(s-a)^{n+1}}, \qquad s > a$
$u_c(t)$	$\dfrac{e^{-cs}}{s}, \qquad s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$f(ct)$	$\dfrac{1}{c}\,F\left(\dfrac{s}{c}\right), \qquad c > 0$
$f \star g = \int_0^t f(t-u)g(u)\,du$	$F(s)\cdot G(s)$
$\delta(t-c)g(t)$	$e^{-cs}g(c)$
$y^{(n)}(t)$	$-y^{(n-1)} - \ldots - s^{n-1}y(0) + s^nY(s)$
$(-t)^n f(t)$	$F^{(n)}(s)$

Example. Compute $\mathcal{L}\left(3\cos 2t - t^4\right)$.

$$\mathcal{L}\left(3\cos 2t - t^4\right) = \mathcal{L}\left(3\cos 2t\right) - \mathcal{L}\left(t^4\right) = \frac{3s}{s^2+4} - \frac{4!}{s^5}. \quad \square$$

Example. (The exponential shift) Compute $\mathcal{L}\left(e^{3t}\sinh 5t\right)$.

I'll use the **exponential shift** formula

$$\mathcal{L}\left(e^{ct}f(t)\right) = F(s-c).$$

e^{ct} corresponds to e^{3t}, so $c = 3$. In addition, $f(t) = \sinh 5t$. Since the transform of $f(t)$ is $F(s)$ (by definition),

$$F(s) = \mathcal{L}\left(\sinh 5t\right) = \frac{5}{s^2 - 25}.$$

The formula requires $F(s-c)$, or $F(s-3)$, since $c = 3$. Therefore,

$$\mathcal{L}\left(e^{3t}\sinh 5t\right) = \frac{5}{(s-3)^2 - 25}. \quad \square$$

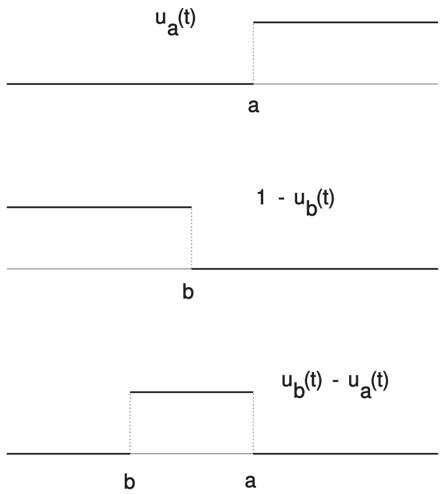
Example. (Step functions) Find $\mathcal{L}\left(h(t)\right)$, where

$$h(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } 0 \le t < 2 \\ t^2 & \text{if } 2 \le t < 3 \\ 2t+1 & \text{if } t \ge 3 \end{cases}$$

I need to express $h(t)$ in terms of step functions $u_a(t)$ so I can apply the step function transform

$$\mathcal{L}\left(u_c(t)f(t-c)\right) = e^{-cs}F(s).$$

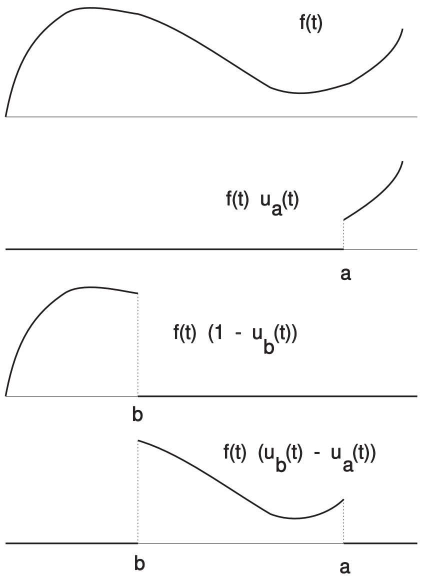
I'll use three kinds of step function combinations:



Each step is 1 unit high.

I'll refer to them, respectively, as a right-hand step, a bump (or pulse), and a left-hand step.

Each kind of step is 1 on a certain interval on a certain interval and 0 elsewhere. Since $1 \cdot \text{junk} = \text{junk}$ and $0 \cdot \text{junk} = 0$, multiplying a function by one of the steps will "cut the function" off except where the step is 1, where it will leave the function unchanged.



To express a function like $h(t)$ in terms of step functions, multiply the "pieces" of $h(t)$ by the corresponding step.

(a) $h(t) = 0$ for $t < 0$. This is a "left-hand piece", so I multiply it by the left-hand step at 0:

$$0 \cdot (1 - u_0(t)) = 0.$$

(b) $h(t) = 1$ for $0 \le t < 2$. This is a "middle piece", so I multiply it by a bump from 0 to 2:

$$1 \cdot (u_0(t) - u_2(t)) = u_0(t) - u_2(t).$$

(c) $h(t) = t^2$ for $2 \le t < 3$. This is a "middle piece", so I multiply it by a bump from 2 to 3:

$$t^2 \cdot (u_2(t) - u_3(t)) = t^2 u_2(t) - t^2 u_3(t).$$

(d) $h(t) = 2t + 1$ for $t \ge 3$. This is a "right-hand piece", so I multiply it by the right-hand step at 3:

$$(2t + 1) \cdot u_3(t).$$

Adding up the pieces,

$$h(t) = 0 + u_0(t) - u_2(t) + t^2 u_2(t) - t^2 u_3(t) + (2t + 1) \cdot u_3(t) =$$

$$u_0(t) + (t^2 - 1) u_2(t) + (1 + 2t - t^2) u_3(t).$$

Next, apply the step function transform formula

$$\mathcal{L}\left(u_c(t) f(t - c)\right) = e^{-cs} F(s).$$

For the first term $u_0(t)$, $f(t-c)=1$, so $f(t)=1$. $F(s)$ is the transform of $f(t)$, which is $\mathcal{L}(1)=\dfrac{1}{s}$. Since $c=0$, the exponential is $e^{-cs}=1$. Therefore,

$$\mathcal{L}\left(u_0(t)\right)=\frac{1}{s}.$$

For $(t^2-1)u_2(t)$, $f(t-c)=t^2-1$. *The standard mistake here is to transform* t^2-1*!* However, the formula has $F(s)$ on the right, and $F(s)$ is the transform of $f(t)$, not of $f(t-c)$.

Instead, notice that the "$u_2(t)$" means that $c=2$. Therefore, $f(t-2)=t^2-1$. What is $f(t)$? You can get $f(t)$ from $f(t-2)$ by replacing t with $t+2$:

$$f(t)=f\left[(t+2)-2\right]=(t+2)^2-1=t^2+4t+3.$$

Therefore,

$$F(s)=\mathcal{L}\left(t^2+4t+3\right)=\frac{2}{s^3}+\frac{4}{s^2}+\frac{3}{s}.$$

Hence,

$$\mathcal{L}\left((t^2-1)u_2(t)\right)=e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{3}{s}\right).$$

The same approach works for $\mathcal{L}\left((1+2t-t^2)u_3(t)\right)$. Here $c=3$, so $f(t-3)=1+2t-t^2$. Replace t with $t+3$:

$$f(t)=f\left[(t+3)-3\right]=1+2(t+3)-(t+3)^2=-t^2-4t-2.$$

Hence,

$$F(s)=\mathcal{L}\left(-t^2-4t-2\right)=-\frac{2}{s^3}-\frac{4}{s^2}-\frac{2}{s}.$$

Finally,

$$\mathcal{L}\left((1+2t-t^2)u_3(t)\right)=e^{-3s}\left(-\frac{2}{s^3}-\frac{4}{s^2}-\frac{2}{s}\right).$$

Putting the pieces together,

$$\mathcal{L}\left(h(t)\right)=\frac{1}{s}+e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{3}{s}\right)+e^{-3s}\left(-\frac{2}{s^3}-\frac{4}{s^2}-\frac{2}{s}\right).\ \Box$$

Example. Find $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+8s+17}\right)$.

The expression is similar to the expression for the transform of a cosine, but the presence of the middle term $8s$ suggests an exponential shift. Complete the square:

$$\frac{s}{s^2+8s+17}=\frac{(s+4)-4}{(s+4)^2+1}.$$

The "$s+4$" on the bottom tells me that I need to express everything — including the top — in terms of $s+4$.

If $F(s)=\dfrac{s-4}{s^2+1}$, then

$$F(s+4)=\frac{s}{(s+4)^2+1}.$$

This is an exponential shift with $c=-4$. Therefore,

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+8s+17}\right)=e^{-4t}\cdot(\cos t-4\sin t),$$

since $\mathcal{L}^{-1}\left(\dfrac{s-4}{s^2+1}\right) = \cos t - 4\sin t$. $\square$

Example. (Convolution) Find $\mathcal{L}^{-1}\left(\dfrac{s}{s^2(s^2+9)}\right)$.

Notice that

$$\frac{s}{s^2(s^2+9)} = \frac{1}{s^2}\cdot\frac{s}{s^2+9},$$

and the table contains the inverse transforms of the two factors on the right:

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t, \quad \mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) = \cos 3t.$$

I'll apply the **convolution formula**, which says that

$$\mathcal{L}^{-1}\left(F(s)\cdot G(s)\right) = f \star g = \int_0^t f(t-u)g(u)\,du.$$

In this case, $f(t) = t$ and $g(t) = \cos 3t$. Therefore,

$$\mathcal{L}^{-1}\left(\frac{s}{s^2(s^2+9)}\right) = \int_0^t (t-u)\cos 3u\,du = \left[\frac{1}{3}(t-u)\sin 3u - \frac{1}{9}\cos 3u\right]_0^t = \frac{1}{9}(1-\cos 3t).$$

For comparison, I'll redo the problem using **partial fractions**. In this case, I need the partial fractions decomposition of $\dfrac{s}{s^2(s^2+9)}$.

$$\frac{s}{s^2(s^2+9)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+9},$$

$$s = As(s^2+9) + B(s^2+9) + (Cs+D)s^2.$$

Setting $s = 0$ yields $0 = 9B$, $B = 0$.

$$s = As(s^2+9) + (Cs+D)s^2, \quad \text{or} \quad s = A(s^3+9s) + Cs^3 + Ds^2.$$

Differentiate:

$$1 = A(3s^2+9) + 3Cs^2 + 2Ds.$$

Setting $s = 0$ gives $1 = 9A$, so $A = \dfrac{1}{9}$.

$$1 = \frac{1}{9}(3s^2+9) + 3Cs^2 + 2Ds$$

Differentiate again:

$$0 = \frac{2}{3}s + 6Cs + 2D.$$

Setting $s = 0$ yields $0 = 2D$, or $D = 0$.

$$0 = \frac{2}{3}s + 6Cs.$$

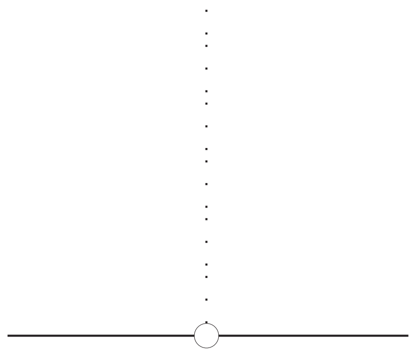
Setting $s = 1$ produces $0 = \dfrac{2}{3} + 6C$, so $C = -\dfrac{1}{9}$.

$$\frac{s}{s^2(s^2+9)} = \frac{1}{9}\,\frac{1}{s} - \frac{1}{9}\,\frac{s}{s^2+9}.$$

The solution is

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{9}\,\frac{1}{s} - \frac{1}{9}\,\frac{s}{s^2+9}\right) = \frac{1}{9}(1 - \cos 3t).\ \square$$

Example. (Impulse functions) $\delta(t-c)$ is the **Dirac delta function**. It is not really a function; it looks like an infinite "spike" at $t = c$:



It makes no sense outside an integral; you can think of it as defined as the limit of a sequence of rectangles with height $\dfrac{1}{h}$ and base h, as $h \to 0$.

$\delta(t-c)$ represents a sudden "shove" (impulse) applied to a system of some kind. Usually, it is scaled by multiplication by another function $g(t)$. Then $\delta(t-c)g(t)$ transforms to $e^{-cs}g(c)$.

For example,

$$\mathcal{L}\left(\delta(t-5)\cos(\pi t)\right) = e^{-5s}\cos(5\pi) = -e^{-5s}.$$

Note that an exponential in s inverse transforms to a step function, rather than an impulse. This makes physical sense: A sudden impulse $\delta(t-c)g(t)$ applied to a system produces effects which start at $t = c$ and last forever, just like a step function $u_c(t)$. $\square$

Example. Solve $y'' - y' - 2y = -e^{-t}$, $y(0) = 0$, $y'(0) = 2$.

Take the transform of both sides of the equation and apply the initial value formula:

$$\mathcal{L}(y'') - \mathcal{L}(y') - 2\,\mathcal{L}(y) = -\,\mathcal{L}\left(e^{-t}\right),$$

$$\left(-2 - 0 + s^2 Y(s)\right) - (0 + sY(s)) - 2Y(s) = -\frac{1}{s+1},$$

$$-2 + (s^2 - s - 2)Y(s) = -\frac{1}{s+1}.$$

As a check on the computation, note that the polynomial $s^2 - s - 2$ that multiplies $Y(s)$ is the same as the operator polynomial $D^2 - D - 2$ that occurs on the left side of the original differential equation.

Now solve for $Y(s)$:

$$Y(s) = -\frac{1}{(s+1)^2(s-2)} + \frac{2}{(s-2)(s+1)} = \frac{2s+1}{(s+1)^2(s-2)}.$$

(Sometimes it is better to do two fractions separately. This is the case, for instance, if one of them can be inverted immediately. In this situation, I didn't recognize either fraction as being on the table. Therefore, I decided to add them so I only have one partial fractions decomposition to do.)

Use partial fractions to expand the right side:

$$\frac{2s+1}{(s+1)^2(s-2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-2},$$

$$2s+1 = A(s+1)(s-2) + B(s-2) + C(s+1)^2.$$

Setting $s = 2$ gives $5 = 9C$, so $C = \frac{5}{9}$. Setting $s = -1$ yields $-1 = -3B$, hence $B = \frac{1}{3}$. Plug these back in:

$$2s+1 = A(s+1)(s-2) + \frac{1}{3}(s-2) + \frac{5}{9}(s+1)^2.$$

Setting $s = 0$ produces $1 = -2A - \frac{2}{3} + \frac{5}{9}$, or $A = -\frac{5}{9}$.
Therefore,

$$Y(s) = \frac{2s+1}{(s+1)^2(s-2)} = -\frac{5}{9}\,\frac{1}{s+1} + \frac{1}{3}\,\frac{1}{(s+1)^2} + \frac{5}{9}\,\frac{1}{s-2}.$$

The solution is

$$y(t) = -\frac{5}{9}e^{-t} + \frac{1}{3}te^{-t} + \frac{5}{9}e^{2t}. \ \square$$

Problems on Laplace Transforms.

1. Compute the Laplace transforms:

(a) $\mathcal{L}\left(t^2 + 4t + 7\right)$

(b) $\mathcal{L}\left(h(t)\right)$, where

$$h(t) = \begin{cases} 1 & \text{if } 0 \le t < 2 \\ t^2 + t & \text{if } t \ge 2 \end{cases}$$

2. Compute the inverse Laplace transforms:

(a) $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+2s+5}\right)$

(b) $\mathcal{L}^{-1}\left(\dfrac{5e^{-3s}}{s^3} - \dfrac{e^{-s}}{s^2}\right)$

(c) $\mathcal{L}^{-1}\left(\dfrac{1}{s^2(s-1)}\right)$

3. Solve for y in terms of t: $y'' + y = h(t)$, $y(0) = 1$, $y'(0) = 0$, and

$$h(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 0 & \text{if } t \ge 1 \end{cases}$$

4. Solve for y in terms of t: $y'' + 2y' + y = 2\delta(t-1)$, $y(0) = 1$, $y'(0) = 1$.

Solutions to the Problems on Laplace Transforms.

1. Compute the Laplace transforms:

(a) $\mathcal{L}\left(t^2 + 4t + 7\right)$

$$\mathcal{L}\left(t^2 + 4t + 7\right) = \frac{2}{s^3} + \frac{4}{s^2} + \frac{7}{s}. \ \square$$

(b) $\mathcal{L}(h(t))$, where

$$h(t) = \begin{cases} 1 & \text{if } 0 \le t < 2 \\ t^2 + t & \text{if } t \ge 2 \end{cases}$$

$$h(t) = 1 \cdot (u_0(t) - u_2(t)) + (t^2 + t)u_2(t) = u_0(t) + (t^2 + t - 1)u_2(t).$$

First,

$$\mathcal{L}(u_0(t)) = \frac{1}{s}.$$

Next, I'll transform $(t^2 + t - 1)u_2(t)$. Here $c = 2$ in the step function formula. This means that $f(t-2) = t^2 + t - 1$.

The step function formula has $F(s)$ on the right, which is the transform of $f(t)$ (as opposed to $f(t-2)$. To find $f(t)$, replace the t's in the equation $f(t-2) = t^2 + t - 1$ with $t + 2$'s. This gives

$$f(t) = (t+2)^2 + (t+2) - 1 = t^2 + 5t + 5.$$

Therefore,

$$\mathcal{L}(f(t))\, F(s) = \frac{2}{s^3} + \frac{5}{s^2} + \frac{5}{s}.$$

Hence,

$$\mathcal{L}\left((t^2 + t - 1)u_2(t)\right) = \left(\frac{2}{s^3} + \frac{5}{s^2} + \frac{5}{s}\right) e^{-2s}.$$

Putting the pieces together,

$$\mathcal{L}(h(t)) = \frac{1}{s} + \left(\frac{2}{s^3} + \frac{5}{s^2} + \frac{5}{s}\right) e^{-2s}. \ \square$$

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2. Compute the inverse Laplace transforms:

(a) $\mathcal{L}^{-1}\left(\dfrac{s}{s^2 + 2s + 5}\right)$

The presence of a middle term in the quadratic suggests completing the square:

$$s^2 + 2s + 5 = (s+1)^2 + 4.$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2 + 2s + 5}\right) = \mathcal{L}^{-1}\left(\frac{s}{(s+1)^2 + 4}\right).$$

Since there is an $s+1$ instead of an s on the bottom, this will be an exponential shift with $c = -1$. Convert everything to $s+1$'s:

$$\frac{s}{(s+1)^2+4} = \frac{s+1-1}{(s+1)^2+4} = \frac{s+1}{(s+1)^2+4} + \frac{1}{(s+1)^2+4}.$$

Now apply the shift to each term.

Since $\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) = \cos 2t$,

$$\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+4}\right) = e^{-t}\cos 2t.$$

Since $\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin 2t$,

$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+4}\right) = e^{-t}\frac{1}{2}\sin 2t2t.$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+2s+5}\right) = e^{-t}\cos 2t - \frac{1}{2}e^{-t}\sin 2t. \ \square$$

(b) $\mathcal{L}^{-1}\left(\frac{5e^{-3s}}{s^3} - \frac{e^{-s}}{s^2}\right)$

$$\mathcal{L}^{-1}\left(\frac{5e^{-3s}}{s^3} - \frac{e^{-s}}{s^2}\right) = \mathcal{L}^{-1}\left(\frac{5e^{-3s}}{s^3}\right) - \mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2}\right).$$

For the first term,

$$\mathcal{L}^{-1}\left(\frac{5}{s^3}\right) = \frac{5}{2}t^2.$$

$\frac{5e^{-3s}}{s^3}$ requires a step function shift. Replace t with $t-3$ and tack on the step function $u_3(t)$:

$$\mathcal{L}^{-1}\left(\frac{5e^{-3s}}{s^3}\right) = \frac{5}{2}(t-3)^2 u_3(t).$$

For the second term,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t.$$

$\frac{e^{-s}}{s^2}$ requires a step function shift with $c = 1$. Replace t with $t-1$ and tack on a $u_1(t)$:

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2}\right) = (t-1)u_1(t).$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{5e^{-3s}}{s^3}\right) - \mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2}\right) = \frac{5}{2}(t-3)^2 u_3(t) - (t-1)u_1(t). \ \square$$

(c) $\mathcal{L}^{-1}\left(\frac{1}{s^2(s-1)}\right)$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s-1)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\cdot\frac{1}{s-1}\right).$$

From the table,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t, \qquad \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t.$$

Therefore, I may apply the convolution formula to obtain

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s-1)}\right) = t \star e^t = \int_0^t (t-u)e^u\,du = \left[(t-u)e^u + e^u\right]_0^t = e^t - t - 1.\ \square$$

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3. Solve for y in terms of t if $y'' + y = h(t)$, $y(0) = 1$, $y'(0) = 0$, and

$$h(t) = \begin{cases} t & \text{if } 0 \le t < 1 \\ 0 & \text{if } t \ge 1 \end{cases}$$

I'll begin by computing the transform of h. First, write h in terms of step functions:

$$h(t) = tu_0(t) - tu_1(t).$$

Next, transform each term on the right. For the first term,

$$\mathcal{L}\left(tu_0(t)\right) = \frac{1}{s^2}.$$

$tu_1(t)$ will be a step function shift with $c = 1$. $f(t-1) = t$, so replacing each t with $t+1$ gives $f(t) = t+1$. Since

$$\mathcal{L}\left(t+1\right) = \frac{1}{s^2} + \frac{1}{s},$$

to follows that

$$\mathcal{L}\left(tu_1(t)\right) == \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s}.$$

Therefore, the transform of the original equation is

$$\left(-0 - s + s^2Y(s)\right) + Y(s) = \frac{1}{s^2} - \left(\frac{1}{s^2} + \frac{1}{s}\right)e^{-s}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{s}{s^2+1} + \frac{1}{s^2(s^2+1)} - \frac{1}{s^2(s^2+1)}e^{-s} + \frac{1}{s(s^2+1)}e^{-s}.$$

To find $y(t)$, I'll invert each term separately.
First,

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = \cos t.$$

Next, observe that $\frac{1}{s^2(s^2+1)}$ occurs in the second and third terms. By partial fractions,

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}.$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}\right) = t - \sin t.$$

This finishes the second term; the third term is $\dfrac{1}{s^2(s^2+1)}e^{-s}$, a step function shift with $c = 1$. Take the $t - \sin t$ from the $\dfrac{1}{s^2(s^2+1)}$, replace the t's with $t-1$'s, and tack on $u_1(t)$:

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}e^{-s}\right) = u_1(t)\left((t-1)\sin(t-1)\right).$$

For the fourth term, I need to decompose $\dfrac{1}{s(s^2+1)}$. I could use partial fractions; an alternative is to do some algebra:

$$\frac{1}{s(s^2+1)} = \frac{s^2+1-s^2}{s(s^2+1)} = \frac{s^2+1}{s(s^2+1)} - \frac{s^2}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}.$$

Then

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) = 1 - \cos t.$$

The shift is on: e^{-s} tells me to replace $1-\cos t$ with $1-\cos(t-1)$ and tack on a $u_1(t)$:

$$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}e^{-s}\right) = u_1(t)\left(1-\cos(t-1)\right).$$

Hence,

$$y(t) = \cos t + t - \sin t - \left((t-1) - \sin(t-1) + 1 - \cos(t-1)\right)u_1(t). \ \square$$

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4. Solve for y in terms of t: $y'' + 2y' + y = 2\delta(t-1)$, $y(0) = 1$, $y'(0) = 1$.

Transform both sides:

$$\left(-1 - s + s^2 Y(s)\right) + 2\left(-1 + sY(s)\right) + Y(s) = 2e^{-s}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{s+3}{(s+1)^2} + \frac{2}{(s+1)^2}e^{-s}.$$

For the first term, write

$$\frac{s+3}{(s+1)^2} = \frac{s+1+2}{(s+1)^2} = \frac{1}{s+1} + \frac{2}{(s+1)^2}.$$

By the table,

$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}$$

For $\dfrac{2}{(s+1)^2}$, the table yields

$$\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2}\right) = 2te^{-t}.$$

Conveniently, $\dfrac{2}{(s+1)^2}$ appears in the remaining term $\dfrac{2}{(s+1)^2}e^{-s}$. This is a step function shift with $c = 1$. Replace t with $t-1$ in $2te^{-t}$, and tack on a $u_1(t)$:

$$\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2}e^{-s}\right) = 2u_1(t)(t-1)e^{-(t-1)}.$$

The answer is

$$y(t) = e^{-t} + 2te^{-t} + 2u_1(t)(t-1)e^{-(t-1)}. \ \square$$