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The inverse of a matrix.

Proposition. Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix. The rows of AB are linear combinations of the rows of B.

Proof. Here's a picture which shows what happens with the first row of AB:

$$\begin{bmatrix} \leftarrow & a_1 & \rightarrow \\ & \vdots & \\ & & \end{bmatrix} \begin{bmatrix} \leftarrow & b_1 & \rightarrow \\ & b_2 & \\ & \vdots & \\ & b_p & \end{bmatrix} = \begin{bmatrix} \leftarrow & (ab)_1 & \rightarrow \\ & \vdots & \\ & & \end{bmatrix}.$$

Let $(ab)_i$ denote the i-th row of AB. Let b_j denote the j-th row of B. Then

$$(ab)_i = \sum_j a_{ij} b_j . \ \Box$$

Corollary. $\operatorname{rank}(AB) \leq \operatorname{rank} B$.

Proof. The proposition shows that the row space of AB is contained in the row space of B. $\Box$

Definition. An **elementary matrix** is a matrix which represents an elementary row operation. ("Represents" means that multiplying on the left by the elementary matrix performs the row operation.)

$$\begin{array}{c|c} & \begin{array}{ccccc} & i & & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & 1 & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array}$$

interchanges rows i and j.

$$\begin{array}{cc} & \begin{array}{ccc} & i & \end{array} \\ \begin{array}{c} \\ i \\ \\ \end{array} & \begin{pmatrix} & \vdots & \\ \cdots & a & \cdots \\ & \vdots & \end{pmatrix} \end{array}$$

multiplies row i by a.

$$\begin{array}{cc} & \begin{array}{ccccc} & i & & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & 1 & \cdots & a & \cdots \\ & \vdots & & \vdots & \\ \cdots & 0 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array}$$

replaces row i with row i plus a times row j.

Their inverses are the elementary matrices

$$\begin{array}{cc} & \begin{array}{ccccc} & i & & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & 0 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \\ \cdots & 1 & \cdots & 0 & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{ccc} & i & \end{array} \\ \begin{array}{c} \\ i \\ \\ \end{array} & \begin{pmatrix} & \vdots & \\ \cdots & \frac{1}{a} & \cdots \\ & \vdots & \end{pmatrix} \end{array}, \quad \begin{array}{cc} & \begin{array}{ccccc} & i & & j & \end{array} \\ \begin{array}{c} \\ i \\ \\ j \\ \\ \end{array} & \begin{pmatrix} & \vdots & & \vdots & \\ \cdots & 1 & \cdots & -a & \cdots \\ & \vdots & & \vdots & \\ \cdots & 0 & \cdots & 1 & \cdots \\ & \vdots & & \vdots & \end{pmatrix} \end{array},$$

respectively.

Definition. An $n \times n$ matrix A is **invertible** if there is an $n \times n$ matrix B such that $AB = BA = I$, where I is the $n \times n$ identity matrix.

Theorem. Let A be an $n \times n$ matrix. The following are equivalent:

1. A is row equivalent to I.
2. A is a product of elementary matrices.
3. A is invertible.

Proof. $(1 \Rightarrow 2)$ Let $E_1, \ldots, E_p$ be elementary matrices which row reduce A to I:

$$E_1 \cdots E_p A = I.$$

Then

$$A = E_p^{-1} \cdots E_1^{-1}.$$

Since the inverse of an elementary matrix is an elementary matrix, A is a product of elementary matrices.

$(2 \Rightarrow 3)$ Write A as a product of elementary matrices:

$$A = F_1 \cdots F_q.$$

Now

$$F_1 \cdots F_q \cdot F_q^{-1} \cdots F_1^{-1} = I,$$

$$F_q^{-1} \cdots F_1^{-1} \cdot F_1 \cdots F_q = I.$$

Hence,

$$A^{-1} = F_q^{-1} \cdots F_1^{-1}.$$

$(3 \Rightarrow 1)$ Suppose A is invertible. Then

$$\operatorname{rank}\left(A^{-1}A\right) \leq \operatorname{rank} A,$$

$$\operatorname{rank} I \leq \operatorname{rank} A,$$

$$n \leq \operatorname{rank} A.$$

Since A is an $n \times n$ matrix, $\operatorname{rank} A \leq n$. Hence, $\operatorname{rank} A = n$.

Row reduce A to a row reduced echelon matrix R. Since $\operatorname{rank} R = n$, R has n leading coefficients, so $R = I$. □

Algorithm. The proof provides an algorithm for inverting a matrix A.

If $E_1, \ldots, E_p$ are elementary matrices which row reduce A to I, then

$$E_1 \cdots E_p A = I.$$

Then

$$A = E_p^{-1} \cdots E_1^{-1} \quad \text{so} \quad A^{-1} = E_1 \cdots E_p \cdot I.$$

That is, the row operations which reduce A to the identity also transform the identity into A^{-1}.

Example. To invert

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 3 \\ 0 & 1 & 1 \end{bmatrix},$$

form the **augmented matrix**

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right].$$

Next, row reduce the augmented matrix. The row operations are entirely determined by the block on the left, which is the original matrix. The row operations turn the left block into the identity, while simultaneously turning the identity on the right into the inverse.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \quad \to \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 3 & -5 \\ 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right].$$

Thus,

$$A^{-1} = \begin{bmatrix} 4 & 3 & -5 \\ -1 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}. \square$$