4-29-1994

Final Review - Part 1.

1. Determine all isomorphism classes of abelian groups of order $2^3 \cdot 3^3$. For each isomorphism class, give the primary decomposition and the corresponding invariant factor decomposition.

$$\begin{aligned} 2^3: &\quad 2^3, 2\cdot 2^2, 2\cdot 2\cdot 2 \\ 3^3: &\quad 3^3, 3\cdot 3^2, 3\cdot 3\cdot 3 \end{aligned}$$

Primary decomposition	Invariant factor decomposition
$\mathbb{Z}_8 \times \mathbb{Z}_{27}$	$\mathbb{Z}_{216}$
$\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_9$	$\mathbb{Z}_3 \times \mathbb{Z}_{72}$
$\mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{24}$
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_{27}$	$\mathbb{Z}_2 \times \mathbb{Z}_{108}$
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_9$	$\mathbb{Z}_6 \times \mathbb{Z}_{36}$
$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_3 \times \mathbb{Z}_6 \times \mathbb{Z}_{12}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{27}$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{54}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_9$	$\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{18}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	$\mathbb{Z}_6 \times \mathbb{Z}_6 \times \mathbb{Z}_6$

$\square$

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2. Construct a 2×2 real matrix A such that A has a real eigenvalue but A is not diagonalizable.

Let
$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

The only eigenvalue of A is 1. Now
$$A - I = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$
so eigenvectors have the form $\langle a, 0\rangle$. There is at most one independent eigenvector, so A is not diagonalizable. $\square$

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3. Prove or disprove: The ideal $\langle x^2 - 1\rangle$ in $\mathbb{R}[x]$ is a prime ideal.

$(x-1)(x+1) \in \langle x^2 - 1\rangle$, but neither $x - 1$ nor $x + 1$ is in the ideal (since every nonzero element of $\langle x^2 - 1\rangle$ is divisible by $x^2 - 1$, and must therefore have degree ≥ 2). Hence, $\langle x^2 - 1\rangle$ is not prime. $\square$

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4. Consider the ideal
$$I = \langle x^2 - 9, x^2 - 2x - 3\rangle$$
in $\mathbb{R}[x]$. Find a polynomial $f(x) \in \mathbb{R}[x]$ such that $I = \langle f(x)\rangle$. (Prove that your f works.)

The ideal will be generated by a greatest common divisor of $x^2 - 9$ and $x^2 - 2x - 3$. Since $x^2 - 9 = (x-3)(x+3)$ and $x^2 - 2x - 3 = (x-3)(x+1)$, I can use $x - 3$. Clearly, $\langle x^2 - 9, x^2 - 2x - 3\rangle \subset \langle x - 3\rangle$.

Apply the Euclidean algorithm:

$$x^2 - 9 = (1)(x^2 - 2x - 3) + (2x - 6)$$

$$x^2 - 2x - 3 = \left(\frac{1}{2}x + \frac{1}{2}x\right)(2x - 6) + 0$$

From the first equation,

$$x - 3 = \frac{1}{2}(x^2 - 9) - \frac{1}{2}(x^2 - 2x - 3).$$

So if $f(x)(x-3) \in \langle x - 3\rangle$, then

$$f(x)(x-3) = \frac{1}{2}(x^2 - 9)f(x) - \frac{1}{2}(x^2 - 2x - 3)f(x) \in \langle x^2 - 9, x^2 - 2x - 3\rangle.$$

This proves that $\langle x - 3\rangle \subset \langle x^2 - 9, x^2 - 2x - 3\rangle$, so in fact $\langle x - 3\rangle = \langle x^2 - 9, x^2 - 2x - 3\rangle$. □

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5. The matrix of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ relative to the standard basis (on the domain and the range) is

$$[T]_{\text{std,std}} = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}.$$

Find the matrix $[T]_{\mathcal{B},\mathcal{B}}$ of T relative to the basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}.$$

The matrix which translates $\mathcal{B}$ to the standard basis is

$$[\mathcal{B} \to \text{std}] = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}.$$

Hence,

$$[\text{std} \to \mathcal{B}] = [\mathcal{B} \to \text{std}]^{-1} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix}.$$

Therefore,

$$[T]_{\mathcal{B},\mathcal{B}} = [\text{std} \to \mathcal{B}][T]_{\text{std,std}}[\mathcal{B} \to \text{std}] = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}. \quad □$$

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6. Find bases for the row space, the column space, and the null space of the real matrix

$$\begin{bmatrix} 0 & 0 & 1 & 3 \\ 1 & -2 & 0 & -1 \\ 1 & -2 & 2 & 5 \end{bmatrix}.$$

Row reduce:

$$\begin{bmatrix} 0 & 0 & 1 & 3 \\ 1 & -2 & 0 & -1 \\ 1 & -2 & 2 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\langle 1, -2, 0, 1\rangle$ and $\langle 0, 0, 1, 3\rangle$ form a basis for the row space.

The leading coefficients are in the first and third columns. Hence, $\langle 1, 1, 0\rangle$ and $\langle 2, 0, 1\rangle$ form a basis for the column space.

If $\langle a, b, c, d\rangle$ is in the null space, then

$$\begin{array}{rcrcrcrcl} a & - & 2b & & & - & d & = & 0 \\ & & & & c & + & 3d & = & 0 \end{array}$$

So $a = 2b - d$, $c = -3d$, and

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2b - d \\ b \\ -3d \\ d \end{bmatrix} = b \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \cdot \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}.$$

Hence, $\langle 2, 1, 0, 0\rangle$ and $\langle -1, 0, -3, 1\rangle$ form a basis for the null space. $\square$

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7. V is a real inner product space, and $\{v_1, v_2, v_3, v_4\}$ is an *independent* set in V. (Warning: $\dim V$ is not given!) Further, $\{v_1, v_2, v_3\}$ is an *orthonormal* set. Find $w \in V$ (in terms of v_1, v_2, v_3, v_4) such that $\{v_1, v_2, v_3, w\}$ is an orthonormal set.

Just apply Gram-Schmidt. If $(\cdot, \cdot)$ denotes the inner product, take

$$w = v_4 - \sum_{i=1}^{3} (w, v_i) v_i. \ \square$$

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8. R is an integral domain, but R is not a field. Prove that $R[x]$ is not a PID.

Let $a \in R$ such that a is nonzero and a is not invertible. Consider the ideal $I = \langle x - a, x\rangle \subset R[x]$. I claim this ideal is not principal.

Suppose on the contrary that $\langle h(x)\rangle = \langle x - a, x\rangle$. Observe that $a = x - (x - a) \in I$. Hence, $a = h(x)f(x)$ for some $f(x) \in R[x]$. Since $a \neq 0$, h must be a nonzero constant polynomial — say $h(x) = c$.

Since $x \in I$, $c \cdot f(x) = x$ for some $f \in R[x]$. By considering degrees, it's clear that $f(x) = px + q$ for some $p, q \in R$. Now $cpx + cq = x$, so $q = 0$ (since R has no zero divisors) and $cp = 1$.

The last equation implies that c is a unit.

On the other hand, $c \in \langle x - a, x\rangle$, so

$$c = (x - a)f(x) + xg(x).$$

This is an identity, so I may set $x = 0$, which yields $c = -f(0) \cdot a$. Multiply both sides by c^{-1}; I get $1 = -c^{-1} f(0) \cdot a$, which implies that a is a unit $\times$.

Hence, the ideal is not principal. $\square$