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Finitely generated abelian groups.

There is no (known) formula which gives the number of groups of order n for any $n > 0$. However, it's possible to classify the finite *abelian* groups of order n. This classification follows from the **structure theorem for finitely generated abelian groups**.

Definition. Let G be an abelian group. The **torsion subgroup** of G is

$$T = \{g \in G \mid ng = 0 \text{ for some } n \in \mathbb{Z}^+\}.$$

I'd better check that the definition makes sense!

Lemma. Let G be an abelian group. The torsion subgroup of G is a subgroup of G.

Proof. Let T be the torsion subgroup of G. $0 \in T$, so T is nonempty. Let $a, b \in T$. I must show $a - b \in T$. Find positive integers m, n, such that $ma = 0$ and $nb = 0$. Then

$$mn(a - b) = mna - mnb = 0 - 0 = 0.$$

Therefore, $a - b \in T$, and $T < G$. □

Definition. A group G is **torsion free** if the only element of finite order is the identity.

Lemma. Let G be an abelian group, T the torsion subgroup of G. Then G/T is torsion free.

Proof. Suppose $g + T$ is an element of G/T of finite order $n > 0$ and $g + T$ is not the identity of G/T. Now $n(g + T) = T \in G/T$, i.e. $ng \in T$. Thus, there is a positive integer m such that $m(ng) = 0$. Since $mn > 0$, this implies that $g \in T$, so $g + T = T \bigtimes$.

It follows that there is no element of finite order besides the identity, and therefore G/T is torsion free. □

In fact, if G is a finitely generated abelian group, more is true.

Definition. An abelian group G is **finitely generated** if there are elements $x_1, \ldots, x_n \in G$ such that every element $x \in G$ can be written as

$$x = a_1 a_1 + \cdots + a_n x_n, \qquad a_i \in \mathbb{Z}.$$

Note that this expression need not be unique.

Definition. A **free abelian group** is a direct sum of copies of $\mathbb{Z}$ (possibly infinitely many copies).

The number of copies (in the sense of cardinality) is the **rank** of the free abelian group. It's possible to prove that the rank of a free abelian group is well-defined.

Theorem. Let G be a finitely generated abelian group.

1. $G = T \times F$, where T is the torsion subgroup and F is a free abelian group.
2. The rank of F is uniquely determined by G.
3. The torsion part T can be written as a direct sum of cyclic groups in the following ways. Each decomposition is unique (in the first case, up to the order of the factors):

$$T \approx \mathbb{Z}_{{p_1}^{r_1}} \times \mathbb{Z}_{{p_2}^{r_2}} \times \cdots \times \mathbb{Z}_{{p_n}^{r_n}}$$

$$T \approx \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \cdots \times \mathbb{Z}_{d_m}, \qquad 1 \leq d_1 \mid d_2 \mid \cdots \mid d_m$$

In the first case, the p's are primes (not necessarily distinct), and $r_i > 0$ for all i. The first case is called a **primary decomposition** while the second case is called an **invariant factor decomposition**.

Example. Find the invariant factor decomposition and the primary decomposition for every abelian group of order 720.

$720 = 2^4 \cdot 3^2 \cdot 5$. I write down all possible ways of factoring each prime power:

$$2^4: \quad 2^4, 2 \cdot 2^3, 2^2 \cdot 2^2, 2 \cdot 2 \cdot 2^2, 2 \cdot 2 \cdot 2 \cdot 2$$

$$3^2: \quad 3^2, 3 \cdot 3$$

$$5: \quad 5$$

I write down the primary decompositions by grabbing one of the 2^4 factorizations, one of the 3^2 factorizations, and the lone 5. I'll give the corresponding invariant factor decompositions, and tell you below how to get them.

Primary decomposition	Invariant factor decomposition
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{30}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{90}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$	$\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{60}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{180}$
$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$	$\mathbb{Z}_{12} \times \mathbb{Z}_{60}$
$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5$	$\mathbb{Z}_4 \times \mathbb{Z}_{180}$
$\mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$	$\mathbb{Z}_6 \times \mathbb{Z}_{120}$
$\mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_5$	$\mathbb{Z}_2 \times \mathbb{Z}_{360}$
$\mathbb{Z}_{16} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$	$\mathbb{Z}_3 \times \mathbb{Z}_{240}$
$\mathbb{Z}_{16} \times \mathbb{Z}_9 \times \mathbb{Z}_5$	$\mathbb{Z}_{720}$

Here's how to get the invariant factor decomposition from the primary decomposition:

$$\begin{array}{cccc} 2 & 2 & 2 & 2 \\ & & 3 & 3 \\ & & & 5 \\ \hline 2 & 2 & 6 & 30 \end{array}$$

I wrote the factors for each prime in three rows, all right justified. Then I multiplied the numbers in each column. The products 2, 2, 6, and 30 are the invariant factors.

You can go the other way, too. For example, to get the primary decomposition for $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_{60}$, write down the prime factorization for each invariant factor: $2 = 2$, $6 = 2 \cdot 3$, and $60 = 2^2 \cdot 3 \cdot 5$. The primary decomposition is $2 \cdot 2 \cdot 2^2 \cdot 3 \cdot 3 \cdot 5$. □

Example. Consider the subgroup H of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ consisting of all x, y, z such that

$$\begin{array}{ccccccc} x & + & y & + & 2z & = & 0 \\ & & y & - & z & = & 0 \end{array}$$

(a) Identify H as a product of cyclic groups.

Rewrite the equations as

$$\begin{array}{rcrcrcl} x & + & & + & 3z & = & 0 \\ & & y & - & z & = & 0 \end{array}$$

Then $x = -3z$ and $y = z$, so $\langle x, y, z \rangle = \langle -3z, z, z \rangle$. It follows that

$$H = \langle (-3, 1, 1) \rangle \,. \square$$

(b) Identify the quotient $\dfrac{\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}}{H}$ as a product of cyclic groups.

Note that $H \approx \mathbb{Z}$, but you can't "cancel" a factor of $\mathbb{Z}$ from the top and bottom. For instance, $\mathbb{Z}/3\mathbb{Z} \approx \mathbb{Z}_3$, even though $3\mathbb{Z} \approx \mathbb{Z}$.

Define $f : \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ by

$$f\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 3z \\ y - z \end{bmatrix} .$$

This is clearly a group homomorphism, and $\ker f = H$. If $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, then

$$f\left(\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} .$$

Hence, f is surjective.

By the First Isomorphism Theorem,

$$\frac{\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}}{H} = \frac{\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}}{\ker f} \approx \operatorname{im} h = \mathbb{Z} \times \mathbb{Z}. \square$$

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