Math 228
4-1-1994

Solutions to Problem Set 25.

[page 364]

2. Express the general solution of the system

$$\vec{x}\,' = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \vec{x}$$

in terms of real-valued functions. Also sketch a few of the trajectories and describe the behavior of the solutions as $t \to \infty$.

$$A = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}.$$

The characteristic polynomial is $5 + 2\lambda + \lambda^2$; the eigenvalues are $\lambda = -1 \pm 2i$.

Consider the eigenvalue $\lambda = -1 + 2i$:

$$A - (-1 + 2i)I = \begin{bmatrix} -2i & -4 \\ 1 & -2i \end{bmatrix}$$

In a 2×2 matrix $A - \lambda I$, the two rows must be multiples of one another. Hence, I can ignore the second row. An eigenvector $\langle a, b\rangle$ must satisfy

$$\begin{bmatrix} -2i & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

That is,

$$(-2i) \cdot a + (-4) \cdot b = 0.$$

There is an obvious way to solve this equation: Swap $-2i$ and 4 and negate one of them (e.g. change $-2i$ to $2i$). Then

$$(-2i) \cdot (-4) + (-4) \cdot (2i) = 0.$$

Thus, $a = -4$, $b = 2i$ works, and $\langle -4, 2i\rangle$ is an eigenvector.

The solution corresponding to this eigenvector is

$$e^{(-1+2i)t} \begin{bmatrix} -4 \\ 2i \end{bmatrix}.$$

Expand it:

$$e^{(-1+2i)t} \begin{bmatrix} -4 \\ 2i \end{bmatrix} = e^{-t} e^{2it} \begin{bmatrix} -4 \\ 2i \end{bmatrix} = e^{-t}(\cos 2t + i \sin 2t) \begin{bmatrix} -4 \\ 2i \end{bmatrix} = e^{-t} \begin{bmatrix} -4\cos 2t - 4i \sin 2t \\ -2\sin 2t + 2i \cos 2t \end{bmatrix}.$$

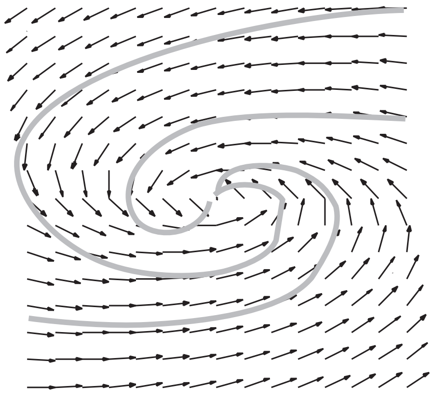
I obtain two independent real solutions (and hence the general solution) by taking the real and imaginary parts:

$$\operatorname{re} e^{(-1+2i)t} \begin{bmatrix} -4 \\ 2i \end{bmatrix} = e^{-t} \begin{bmatrix} -4\cos 2t \\ -2\sin 2t \end{bmatrix},$$

$$\operatorname{im} e^{(-1+2i)t} \begin{bmatrix} -4 \\ 2i \end{bmatrix} = e^{-t} \begin{bmatrix} -4\sin 2t \\ 2\cos 2t \end{bmatrix}.$$

The solution is

$$\vec{x} = c_1 e^{-t} \begin{bmatrix} -4\cos 2t \\ -2\sin 2t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -4\sin 2t \\ 2\cos 2t \end{bmatrix}.$$



As $t \to \infty$, the trajectories approach the origin. □

---

5. Express the general solution of the system

$$\vec{x}\,' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \vec{x}$$

in terms of real-valued functions. Also sketch a few of the trajectories and describe the behavior of the solutions as $t \to \infty$.

$$A = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}.$$

The characteristic polynomial is $2 + 2\lambda + \lambda^2$; the eigenvalues are $\lambda = -1 \pm i$.
Consider the eigenvalue $\lambda = -1 + i$:

$$A - (-1+i)I = \begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix}$$

Following the procedure of Problem 2, I can see that $\langle 1, 2-i \rangle$ is an eigenvector.
The solution corresponding to this eigenvector is

$$e^{(-1+i)t} \begin{bmatrix} 1 \\ 2-i \end{bmatrix}.$$

Expand it:

$$e^{(-1+i)t} \begin{bmatrix} 1 \\ 2-i \end{bmatrix} = e^{-t} e^{it} \begin{bmatrix} 1 \\ 2-i \end{bmatrix} = e^{-t}(\cos t + i \sin t) \begin{bmatrix} 1 \\ 2-i \end{bmatrix} = e^{-t} \begin{bmatrix} \cos t + i \sin t \\ (2\cos t + \sin t) + i(-\cos t + 2\sin t) \end{bmatrix}.$$

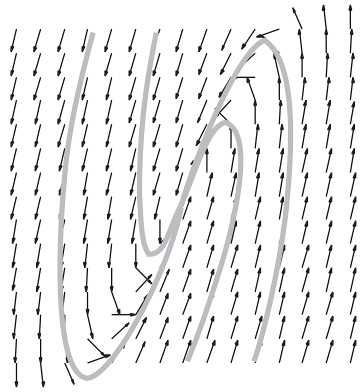
I obtain two independent real solutions (and hence the general solution) by taking the real and imaginary parts:

$$\operatorname{re} e^{(-1+i)t} \begin{bmatrix} 1 \\ 2-i \end{bmatrix} = e^{-t} \begin{bmatrix} \cos t \\ 2\cos t + \sin t \end{bmatrix},$$

$$\operatorname{im} e^{(-1+i)t} \begin{bmatrix} 1 \\ 2-i \end{bmatrix} = e^{-t} \begin{bmatrix} \sin t \\ -\cos t + 2\sin t \end{bmatrix}.$$

The solution is

$$\vec{x} = c_1 e^{-t} \begin{bmatrix} \cos t \\ 2\cos t + \sin t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin t \\ -\cos t + 2\sin t \end{bmatrix}.$$



As $t \to \infty$, the trajectories approach the origin. □

---

7. Express the general solution of the system

$$\vec{x}\,' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} \vec{x}$$

in terms of real-valued functions.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}.$$

The characteristic polynomial is $5 - 7\lambda + 3\lambda^2 - \lambda^3$. The eigenvalues are $\lambda = 1$ and $\lambda = 1 \pm 2i$. First, consider $\lambda = 1$:

$$A - I = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \underset{r_1 \leftrightarrow r_2}{\longrightarrow} \begin{bmatrix} 2 & 0 & -2 \\ 0 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix} \underset{r_1 \to \frac{1}{2} r_1}{\longrightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 3 & 2 & 0 \end{bmatrix} \underset{r_3 \to r_3 - 3r_1}{\longrightarrow}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix} \underset{r_2 \leftrightarrow r_3}{\longrightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \underset{r_2 \to \frac{1}{2} r_2}{\longrightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix says $a - c = 0$, so $a = c$, and $b + \frac{3}{2}c = 0$, so $b = -\frac{3}{2}c$. The eigenvectors are

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} c \\ -\frac{3}{2}c \\ c \end{bmatrix} = c \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}.$$

This yields the solution

$$e^t \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix}.$$

Next, consider $\lambda = 1 + 2i$:

$$A - (1+2i)I = \begin{bmatrix} -2i & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix} \underset{r_1 \to -\frac{1}{2i}r_1}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix} \underset{r_2 \to r_2 - 2r_1}{\longrightarrow}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2i & -2 \\ 3 & 2 & -2i \end{bmatrix} \underset{r_3 \to r_3 - 3r_1}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2i & -2 \\ 0 & 2 & -2i \end{bmatrix} \underset{r_2 \to -\frac{1}{2i}r_2}{\longrightarrow}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & 2 & -2i \end{bmatrix} \underset{r_3 \to r_3 - 2r_2}{\longrightarrow} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix says $a = 0$ and $b - ic = 0$, or $b = ic$. The eigenvectors are

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ ic \\ c \end{bmatrix} = c \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}.$$

Since $1 - 2i$ is the conjugate of $1 + 2i$, I can find an eigenvector for $1 - 2i$ by taking the conjugate of an eigenvector for $1 + 2i$. Thus, $\langle 0, -i, 1 \rangle$ is an eigenvector for $1 - 2i$.

However, I do not need the second eigenvector to obtain two independent real solutions corresponding to $1 \pm 2i$. $\langle 0, i, 1 \rangle$ is an eigenvector for $1 + 2i$, and this gives the solution

$$e^{(1+2i)t} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}.$$

Expand it:

$$e^{(1+2i)t} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = e^t e^{2it} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = e^t(\cos 2t + i \sin 2t) \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = e^t \begin{bmatrix} 0 \\ -\sin 2t + i \cos 2t \\ \cos 2t + i \sin 2t \end{bmatrix}.$$

I can obtain two independent real solutions by taking the real and imaginary parts:

$$\operatorname{re} e^{(1+2i)t} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = e^t \begin{bmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{bmatrix},$$

$$\operatorname{im} e^{(1+2i)t} \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix} = e^t \begin{bmatrix} 0 \\ \cos 2t \\ \sin 2t \end{bmatrix}.$$

Combine these with the solution generated by $\lambda = 1$ to obtain the general solution:

$$\vec{x} = c_1 e^t \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ -\sin 2t \\ \cos 2t \end{bmatrix} + c_3 e^t \begin{bmatrix} 0 \\ \cos 2t \\ \sin 2t \end{bmatrix}. \quad \square$$

---

9. Find the solution of the initial value problem

$$\vec{x}\,' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Describe the behavior of the solution as $t \to \infty$.

$$A = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}.$$

The characteristic polynomial is $2 + 2\lambda + \lambda^2$; the eigenvalues are $\lambda = -1 \pm i$.
Consider the eigenvalue $\lambda = -1 + i$:

$$A - (-1+i)I = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix}$$

As in Problem 2, I see that $\langle 5, 2-i \rangle$ is an eigenvector.
From this, I obtain the solution

$$e^{(-1+i)t} \begin{bmatrix} 5 \\ 2-i \end{bmatrix}.$$

Expand it:

$$e^{(-1+i)t} \begin{bmatrix} 5 \\ 2-i \end{bmatrix} = e^{-t} e^{it} \begin{bmatrix} 5 \\ 2-i \end{bmatrix} = e^{-t}(\cos t + i \sin t) \begin{bmatrix} 5 \\ 2-i \end{bmatrix} = e^{-t} \begin{bmatrix} 5\cos t + 5i \sin t \\ (2\cos t + \sin t) + i(-\cos t + 2\sin t) \end{bmatrix}.$$

I take the real and imaginary parts to obtain two independent real solutions:

$$\operatorname{re} e^{(-1+i)t} \begin{bmatrix} 5 \\ 2-i \end{bmatrix} = e^{-t} \begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix},$$

$$\operatorname{im} e^{(-1+i)t} \begin{bmatrix} 5 \\ 2-i \end{bmatrix} = e^{-t} \begin{bmatrix} 5\sin t \\ -\cos t + 2\sin t \end{bmatrix}.$$

The general solution is

$$\vec{x} = c_1 e^{-t} \begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 5\sin t \\ -\cos t + 2\sin t \end{bmatrix}.$$

Plug in the initial condition:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

Therefore, $c_1 = \dfrac{1}{5}$, $c_2 = -\dfrac{3}{5}$. Therefore, the solution is

$$\vec{x} = \frac{1}{5} e^{-t} \begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix} - \frac{3}{5} e^{-t} \begin{bmatrix} 5\sin t \\ -\cos t + 2\sin t \end{bmatrix} = e^{-t} \begin{bmatrix} \cos t - 3\sin t \\ \cos t - \sin t \end{bmatrix}.$$

As $t \to \infty$, the solution approaches the origin, because $e^{-t} \to 0$. □

"We promise according to our hopes, and perform according to our fears." - La Rochefoucauld