Math 228
2-23-1994

Solutions to Problem Set 15.

1. (a) $y_1 = \sqrt{x}$ is a solution to the homogeneous equation

$$2x^2 y'' - xy' + y = 0.$$

Use reduction of order to show that $y_2 = x$ is a second independent solution. If you have trouble with the algebra, go on to part (b).

Set $y = v\sqrt{x}$, so

$$y' = v'\sqrt{x} + \frac{v}{2\sqrt{x}}, \quad y'' = v''\sqrt{x} + \frac{v'}{\sqrt{x}} - \frac{v}{4x^{3/2}}.$$

Substitute:

$$\begin{aligned} 2x^2 y'' - xy' + y &= 2x^{5/2}v'' + 2x^{3/2}v' - \frac{1}{2}x^{1/2}v \\ &\qquad - x^{3/2}v' - \frac{1}{2}x^{1/2}v \\ &\qquad + x^{1/2}v \\ &= 2x^{5/2}v'' + x^{3/2}v' = 0. \end{aligned}$$

Therefore,

$$v'' + \frac{1}{2}x^{-1}v' = 0.$$

Let $u = v'$, so $u' = v''$:

$$u' + \frac{1}{2}x^{-1}u = 0,$$

$$\int \frac{du}{u} = -\int \frac{1}{2}x^{-1}\,dx,$$

$$\ln|u| = -\frac{1}{2}\ln|x| + c,$$

$$u = \frac{c_0}{\sqrt{x}},$$

$$v' = \frac{c_0}{\sqrt{x}},$$

$$v = 2c_0\sqrt{x} + c_1.$$

Therefore,

$$y = v\sqrt{x} = 2c_0 x + c_1\sqrt{x}.$$

$y_2 = x$ is a second, independent homogeneous solution. $\square$

(b) Use the independent homogeneous solutions $y_1 = \sqrt{x}$ and $y_2 = x$ to solve the equation

$$2x^2 y'' - xy' + y = 2x^3.$$

Be careful to put the equation into standard form before you start! If you have trouble with the algebra, open your window and shout: "Iä! Iä! Cthulhu fhtagn!"

First, I need to put the equation into standard form:

$$y'' - \frac{1}{2x}y' + \frac{1}{2x^2}y = x.$$

Next, set up the variation of parameters equation:

$$\begin{bmatrix} \sqrt{x} & x \\ \dfrac{1}{2\sqrt{x}} & 1 \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix}.$$

Invert the coefficient matrix and put the inverse on the right:

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{2}{\sqrt{x}} \begin{bmatrix} 1 & -x \\ -\dfrac{1}{2\sqrt{x}} & \sqrt{x} \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} = \begin{bmatrix} -2x^{3/2} \\ 2x \end{bmatrix}.$$

Solve for u_1 and u_2:

$$u_1' = -2x^{3/2}, \quad u_1 = -\frac{4}{5}x^{5/2} + c_1,$$

$$u_2' = 2x, \quad u_2 = x^2 + c_2.$$

The solution is

$$y = -\frac{4}{5}x^3 + c_1\sqrt{x} + x^3 + c_2 x = \frac{1}{5}x^3 + c_1\sqrt{x} + c_2 x. \ \square$$

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2. Find the general solution:

$$(D^2 + 4)y = 4\cos 2x.$$

Use undetermined coefficients. The homogeneous solution is

$$y_h = c_1 \cos 2x + c_2 \sin 2x.$$

To find a particular solution, I'll use complex exponentials. $\cos 2x$ is the real part of $e^{2xi} = \cos 2x + i \sin 2x$. So change the equation to

$$(D^2 + 4)y_p = 4e^{2xi}.$$

Now e^{2xi} is killed by $D - 2i$, so apply $D - 2i$ to both sides:

$$(D - 2i)(D^2 + 4)y_p = (D - 2i)(4e^{2xi}) = 0.$$

Now $D^2 + 4 = (D - 2i)(D + 2i)$, so

$$(D - 2i)^2(D + 2i)y_p = 0.$$

Therefore,

$$y_p = Ae^{2xi} + Bxe^{2xi} + Ce^{-2xi}.$$

e^{2xi} and e^{-2xi} are homogeneous solutions. Throwing them out leaves

$$y_p = Bxe^{2xi}.$$

Differentiate:

$$y_p' = Be^{2xi} + 2Bixe^{2xi}, \quad y_p'' = 4Bie^{2xi} - 4Bxe^{2xi}.$$

Substitute:

$$y_p'' + 4y_p = 4Bie^{2xi} - 4Bxe^{2xi} + 4Bxe^{2xi} = 4Bie^{2xi} = 4e^{2xi}.$$

Therefore, $4Bi = 4$, so $B = \dfrac{1}{i} = -i$. Hence,

$$y_p = ixe^{2xi} = ix(\cos 2x + i \sin 2x) = -x \sin 2x + ix \cos 2x.$$

Take the imaginary part to obtain the real particular solution $-x \sin 2x$. The solution to the original equation is

$$y = c_1 \cos 2x + c_2 \sin 2x - x \sin 2x. \square$$

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3. In this problem, you do not have to solve the equations. In each case, say whether the method of undetermined coefficients is applicable or not. If it is applicable, say what form a particular solution y_p should have. y_p should not include any functions which are part of the general homogeneous solution.

(a) $y'' - 2y' + y = \tan x$.

Undetermined coefficients doesn't apply, because $\tan x$ is not killed by a constant coefficient D polynomial. $\square$

(b) $y'' - 2y' + y = \cosh x$.

The equation is

$$(D-1)^2 y = \frac{1}{2}\left(e^x + e^{-x}\right).$$

The right side is killed by $(D-1)(D+1)$, so

$$(D-1)(D+1)(D-1)^2 y_p = (D-1)(D+1)\left[\frac{1}{2}\left(e^x + e^{-x}\right)\right] = 0.$$

Hence,

$$y_p = Ae^x + Bxe^x + Cx^2e^x + De^{-x}.$$

e^x and e^{-x} are homogeneous solutions. Throwing them out leaves

$$y_p = Bxe^x + Cx^2e^x. \square$$

(c) $(D^2 - 3D - 4)y = \sqrt{x}$.

Undetermined coefficients doesn't apply, because $\sqrt{x}$ is not killed by a constant coefficient D polynomial. $\square$

(d) $(D^2 - 9)y = \cos 3x$.

$\cos 3x$ is killed by $D^2 + 9$, so

$$(D^2 + 9)(D^2 - 9)y_p = (D^2 + 9)(\cos 3x) = 0.$$

Therefore,

$$y_p = A \cos 3x + B \sin 3x + Ce^{3x} + De^{-3x}.$$

e^{3x} and e^{-3x} are homogeneous solutions. Throwing them out leaves

$$y_p = A \cos 3x + B \sin 3x. \square$$

(e) $(D^2 + 9)y = \cos 3x$.

$\cos 3x$ is killed by $D^2 + 9$, so

$$(D^2 + 9)(D^2 + 9)y_p = (D^2 + 9)(\cos 3x) = 0.$$

Therefore,

$$y_p = A\cos 3x + B\sin 3x + Cx\cos 3x + Dx\sin 3x.$$

$\cos 3x$ and $\sin 3x$ are homogeneous solutions. Throwing them out leaves

$$y_p = Cx\cos 3x + Dx\sin 3x. \quad \square$$

(f) $y'' - 4y' = 1 + 2x + 3x^2$.

The equation is

$$(D^2 - 4D)y = 1 + 2x + 3x^2.$$

The right side is killed by D^3, so

$$D^3(D^2 - 4D)y_p = D^3(1 + 2x + 3x^2) = 0.$$

Therefore,

$$y_p = A + Bx + Cx^2 + Dx^3 + Ee^{4x}.$$

A and e^{4x} are homogeneous solutions. Throwing them out leaves

$$y_p = Bx + Cx^2 + Dx^3. \quad \square$$

(g) $(D^2 - 2D + 1)y = e^x(2\cos x - 3\sin x)$.

First, I do the problem using real functions.
$e^x \cos x$ and $e^x \sin x$ are killed by $D^2 - 2D + 2$ (because $m^2 - 2m + 2 = 0$ has roots $1 \pm i$). So

$$(D^2 - 2D + 2)(D^2 - 2D + 1)y_p = (D^2 - 2D + 2)\left[e^x(2\cos x - 3\sin x)\right] = 0.$$

Therefore,

$$y_p = Ae^x \cos x + Be^x \sin x + Ce^x + Dxe^x.$$

e^x and xe^x are homogeneous solutions. Throwing them out leaves

$$y_p = Ae^x \cos x + Be^x \sin x.$$

Using complex exponentials in this problem requires some care. $e^x \cos x$ is the real part of $e^{(1+i)x}$ while $e^x \sin x$ is the imaginary part. But I can't do both at once:

$$(D^2 - 2D + 1)y = 2e^{(1+i)x} - 3e^{(1+i)x}. \quad \text{(WRONG!)}$$

Instead, I would have to do these *two* problems separately:

$$(D^2 - 2D + 1)y = 2e^{(1+i)x}. \tag{1}$$

$$(D^2 - 2D + 1)y = -3e^{(1+i)x}. \tag{2}$$

In both cases, $y_p = Ae^{(1+i)x}$. In (1), after I find A I take the *real* part, because the original term was $\cos x$. In (2), after I find A (which will be different than the A in (1)), I take the imaginary part, because the original term was $\sin x$. Finally, I add the results from (1) and (2) to get a particular solution for the original equation. $\square$

"The best way out is always through." Robert Frost