2-24-1994

Laplace transforms.

An **integral transform** is an operator

$$F(s) = \int_a^b K(s,t) f(t)\, dt.$$

The input to the transform is the function $f(t)$; the output is the function $F(s)$. (By convention, small letters denote the inputs to a transform, and the corresponding capital letters denote the corresponding outputs. For example, the transform of $f(t)$ is denoted $F(s)$.) The function $K(s,t)$ is called the **kernel**.

A transform converts one kind of problem into another. Schematically:

$$\begin{array}{ccc} \{\text{original problem}\} & \rightarrow & \{\text{transformed problem}\} \\ & & \downarrow \\ \left\{\begin{array}{c}\text{solution to} \\ \text{original problem}\end{array}\right\} & \leftarrow & \left\{\begin{array}{c}\text{solution to} \\ \text{transformed problem}\end{array}\right\} \end{array}$$

The procedure is to transform the original problem, solve the transformed problem, then use the **inverse transform** to obtain a solution to the original problem.

The **Laplace transform** is defined by

$$F(s) = \mathcal{L}\left(f(t)\right) = \int_0^\infty e^{-st} f(t)\, dt.$$

The symbol $\mathcal{L}\left(f(t)\right)$ denotes the Laplace transform of $f(t)$; by the convention above, $F(s)$ denotes the Laplace transform of $f(t)$.

The integral which defines the transform is an improper integral. An improper integral may **converge** or **diverge**, depending on the integrand. I have to make some assumptions about the functions I'm feeding into the transform in order to ensure convergence.

Definition. A function $f(t)$ is **piecewise continuous** on the interval $[a,b]$ if:

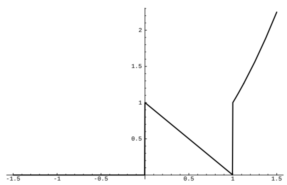
1. The interval may be broken into a finite number of subintervals $a = t_0 < t_1 < \cdots < t_n = b$, such that f is continuous on each subinterval $t_i < t < t_{i+1}$.

2. As t approaches an endpoint of a subinterval from inside the interval, the one-sided limit of $f(t)$ is defined and finite.

Pictorially, condition (1) means that the graph of f on $[a,b]$ is made up of continuous pieces. The second condition implies, for instance, that f has no vertical asymptotes inside the interval.

Example.

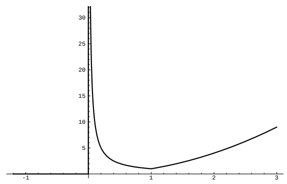
$$f(t) = \begin{cases} 0 & \text{if } -1 \le t < 0 \\ 1 - t & \text{if } 0 \le t < 1 \\ t^2 & \text{if } 1 \le t \le 2 \end{cases}$$



f is piecewise continuous on $[-1, 2]$. □

Example.

$$f(t) = \begin{cases} 0 & \text{if } -1 \le t \le 0 \\ \dfrac{1}{t} & \text{if } 0 < t < 1 \\ t^2 & \text{if } 1 \le t \le 2 \end{cases}$$



f is not piecewise continuous on $[-1, 2]$. □

A function f is of **exponential order** if there are numbers C and a such that

$$|f(t)| \le Ce^{at} \quad \text{for sufficiently large value } t.$$

This means that f does not grow in magnitude faster than an exponential. Heuristically, this should allow the negative exponential in the transform integral to keep the integral from blowing up.

Suppose, then, that f is piecewise continuous on $[0, +\infty)$ (that is, it is piecewise continuous on $[0, m]$ for all $m \ge 0$). Suppose also that f is of exponential order. Then the Laplace transform of $f(t)$

$$F(s) = \mathcal{L}\left(f(t)\right) = \int_0^\infty e^{-st} f(t)\, dt$$

exists for $s > a$.

To see this, break up the integral into two pieces,

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^M e^{-st} f(t)\, dt + \int_M^\infty e^{-st} f(t)\, dt,$$

so that $|f(t)| \le Ce^{at}$ for $t \ge M$.

Piecewise continuity guarantees that the first integral exists.

For the second integral, note that

$$|e^{-st}f(t)| \le e^{-st} \cdot Ce^{at} = Ce^{(a-s)t},$$

because f is of exponential order.

Then

$$\int_0^\infty |e^{-st}f(t)|\,dt \le \int_0^\infty Ce^{(a-s)t}\,dt,$$

and the integral on the right converges provided that $a - s < 0$, i.e. $s > a$. Hence, the same is true for the integral on the left, and likewise for the integral which defines the transform.

Example. Before I actually compute some transforms, here are a couple of trivial but useful observations:

$$\mathcal{L}\left(f(t) + g(t)\right) = \mathcal{L}\left(f(t)\right) + \mathcal{L}\left(g(t)\right)$$

$$\mathcal{L}\left(c \cdot f(t)\right) = c \cdot \mathcal{L}\left(f(t)\right)$$

In words, the transform of a sum is the sum of the transforms, and constants can be pulled out of transforms. These rules are obvious, because they hold for integrals, and transforms are defined by integrals. $\square$

Example. It's important to work out a few transforms by hand to get the idea. However, in doing problems — e.g., on exams — you will be provided with a table of common transforms.

I'll start with something easy — I'll compute $\mathcal{L}\left(t^2\right)$.

$$\mathcal{L}\left(t^2\right) = \int_0^\infty t^2 e^{-st}\,dt = \lim_{p \to \infty} \left[-\frac{t^2}{s}e^{-st} - \frac{2t}{s^2}e^{-st} - \frac{2}{s^3}e^{-st}\right]_0^p =$$

$$\lim_{p \to \infty} \left(\left(-\frac{p^2}{s} - \frac{2p}{s} - \frac{2}{s^3}\right)e^{-sp} + \frac{2}{s^3}\right) = \frac{2}{s^3}.$$

The last step follows because the exponential goes to 0 faster than the powers go to infinity.

Here's the work for the integral:

$$\begin{array}{cccc}
 & \dfrac{d}{dt} & & \int dt \\
+ & t^2 & & e^{-st} \\
 & & \searrow & \\
- & 2t & & -\dfrac{1}{s}e^{-st} \\
 & & \searrow & \\
+ & 2 & & \dfrac{1}{s^2}e^{-st} \\
 & & \searrow & \\
- & 0 & & -\dfrac{1}{s^3}e^{-st}
\end{array}$$

$$\int t^2 e^{-st}\,dt = -\frac{t^2}{s}e^{-st} - \frac{2t}{s^2}e^{-st} - \frac{2}{s^3}e^{-st} + C. \square$$

Example. (The derivative of a transform) Suppose that $F(s) = \mathcal{L}(f(t))$. This means that

$$F(s) = \int_0^\infty f(t) e^{-st}\, dt.$$

Differentiate both sides with respect to s:

$$F'(s) = \int_0^\infty \left[-t f(t)\right] e^{-st}\, dt.$$

(I'm assuming that there's no problem with differentiating under the integral sign. This is okay, because the computation I did above shows that if the transform exists, the integral converges absolutely.) The right side is the transform of $-tf(t)$, so

$$F'(s) = \mathcal{L}\left[-tf(t)\right].$$

For example,

$$\mathcal{L}\left[t^2\right] = \frac{2}{s^3}, \quad \text{so} \quad \mathcal{L}\left[-t \cdot t^2\right] = \frac{d}{ds}\frac{2}{s^3} = -\frac{6}{s^4}.$$

That is,

$$\mathcal{L}\left(t^3\right) = \frac{6}{s^4}.$$

In general, if n is a positive integer,

$$\mathcal{L}\left(t^n\right) = \frac{n!}{s^{n+1}}.$$

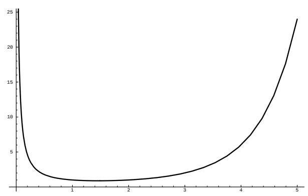
Recall that $n!$, read n-**factorial**, is the product of the integers from 1 to n. For example, $3! = 1 \cdot 2 \cdot 3 = 6$. In fact, the formula above extends to t^a where a is any *real* number greater than -1. The formula in the general case is

$$\mathcal{L}\left(t^a\right) = \frac{\Gamma(a+1)}{s^{a+1}}, \quad s > 0.$$

$\Gamma(a)$ is the **Gamma function**; it is an extension of factorials to real numbers greater than -1, and it is defined by

$$\Gamma(a+1) = \int_0^\infty e^{-x} x^a\, dx.$$

Thus, $\Gamma(a+1) = a!$ when a is a positive integer. Here's a graph of the Gamma function:



If you're using *Mathematica*, its name is `Gamma[]`.

With the Gamma function defined, it's easy to derive the transform above. First,

$$\mathcal{L}\left(t^a\right) = \int_0^\infty e^{-st} t^a\, dt.$$

Let $x = st$, so $dx = s\,dt$, and $t^a = \dfrac{x^a}{s^a}$. When $t = 0$, $x = 0$, and (since $s > 0$) as $t \to \infty$, $x \to \infty$. Therefore,

$$\mathcal{L}\left(t^a\right) = \int_0^\infty e^{-x} \frac{x^a}{s^a} \frac{dx}{s} = \frac{1}{s^{a+1}} \int_0^\infty e^{-x} x^a \, dx = \frac{\Gamma(a+1)}{s^{a+1}}.$$

For example,

$$\mathcal{L}\left(\sqrt{t}\right) = \frac{\Gamma\left(\dfrac{3}{2}\right)}{s^{3/2}}.$$

(It turns out that $\Gamma\left(\dfrac{3}{2}\right) \approx 0.88623$. Better, it turns out that $\Gamma\left(\dfrac{3}{2}\right) = \dfrac{\sqrt{\pi}}{2}$.) $\square$

Example. Compute $\mathcal{L}\left(e^{at} \cos bt\right)$.

I'll use complex exponentials to simplify the computation.

$$e^{at} \cos bt = \operatorname{re}\left(e^{at} \cos bt + ie^{at} \sin bt\right) = \operatorname{re} e^{at} e^{bit} = \operatorname{re} e^{(a+bi)t}.$$

I'll compute the transform of the complex function $e^{(a+bi)t}$, then take the real part at the end.

$$\mathcal{L}\left(e^{(a+bi)t}\right) = \int_0^\infty e^{(a+bi)t} e^{-st} \, dt = \int_0^\infty e^{(a-s+bi)t} \, dt =$$

$$\lim_{p \to \infty} \left[\frac{1}{a-s+bi} e^{(a-s+bi)t}\right]_0^p = \frac{1}{a-s+bi} \lim_{p \to \infty} \left(e^{(a-s+bi)p} - 1\right).$$

I'll wave my hands a bit at the limit. The point is that

$$e^{(a-s+bi)p} = e^{(a-s)p} e^{bip}.$$

The first term goes to 0 provided that $s > a$ (so $a - s < 0$); the second term is $\cos bp + i \sin bp$, which is a complex number of length 1 no matter what p is. Hence, the product goes to 0. Therefore,

$$\mathcal{L}\left(e^{(a+bi)t}\right) = -\frac{1}{a-s+bi} = \frac{1}{s-(a+bi)} = \frac{1}{s-(a+bi)} \frac{s-a+bi}{s-a+bi} = \frac{s-a+bi}{(s-a)^2+b^2}.$$

The transform of $e^{at} \cos bt$ is the real part, so

$$\mathcal{L}\left(e^{at} \cos bt\right) = \frac{s-a}{(s-a)^2+b^2}. \square$$