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Inverse transforms and Initial value problems.

Laplace transforms can be used to solve initial value problems. To do this, I'll need to formula for the transform of a derivative. It follows easily by repeated integration by parts:

$$\begin{array}{cccc}
 & \dfrac{d}{dt} & & \int\, dt \\
+ & e^{-st} & & y^{(n)} \\
 & & \searrow & \\
- & -se^{-st} & & y^{(n-1)} \\
 & & \searrow & \\
+ & s^2 e^{-st} & & y^{(n-2)} \\
 & & \searrow & \\
- & -s^3 e^{-st} & & y^{(n-3)} \\
 & \vdots & & \vdots \\
(-1)^{n-1} & (-1)^{n-1} s^{n-1} e^{-st} & & y' \\
 & & \searrow & \\
(-1)^n & (-1)^n s^n e^{-st} & \longrightarrow & y
\end{array}$$

Thus,

$$\mathcal{L}\left(y^{(n)}(t)\right) = \left[e^{-st} y^{(n-1)} + se^{-st} y^{(n-2)} + \cdots + s^{n-1} e^{-st} y\right]_0^{+\infty} + \int_0^{+\infty} s^n e^{-st} y(t)\, dt =$$

$$\left(-y^{(n-1)}(0) - sy^{(n-2)}(0) - \cdots - s^{n-1} y(0)\right) + s^n\, \mathcal{L}\left(y(t)\right).$$

Here is a typical initial value problem:

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x).$$

The initial conditions are given at $t = 0$: that is, values of $y(0)$, $y'(0)$, ..., $y^{(n-1)}(0)$ are specified.

To solve, I transform the differential equation and use the formula for the transform of a derivative. Let $Y(s) = \mathcal{L}\left(y(t)\right) = \mathcal{L}(y)$ denote the transform of the solution y. I solve the resulting equation for $Y(s)$ and try to **invert** the transform to find y:

$$y(t) = \mathcal{L}^{-1}\left(Y(s)\right).$$

Example. $y'' + y = 1$, $y(0) = 2$, $y'(0) = 0$.

You could solve this equation using undetermined coefficients or variation of parameters; I'll use Laplace transforms to illustrate the method.

Take the transform of both sides, letting $Y(s)$ denote $\mathcal{L}(y)$:

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(1), \quad \left(0 - 2s + s^2 Y(s)\right) + Y(s) = \frac{1}{s}.$$

Simplify and solve for $Y(s)$:

$$Y(s) = \frac{1 + 2s^2}{s(s^2 + 1)}.$$

I used a table to find $\mathcal{L}(1)$. I'll usually refer to a table of transforms to compute the transforms I need.

Next, I need to find a function y such that $\mathcal{L}(y) = Y$. This means $\mathcal{L}^{-1}(Y) = y$, i.e.

$$y = \mathcal{L}^{-1}\left(\frac{1+2s^2}{s(s^2+1)}\right).$$

In order to use our table, I must decompose the right side into forms I recognize. Use **partial fractions**:

$$\frac{1+2s^2}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1},$$

$$1+2s^2 = A(s^2+1) + (Bs+C)s.$$

Plugging in $s=0$ gives $A=1$, so

$$1+2s^2 + (s^2+1) + (Bs+C)s.$$

There are many ways to proceed at this point. For example, you can multiply out the right side and compare coefficients of powers of s on the left and right. Here's an alternative. Differentiate both sides and simplify:

$$4s = 2s + (Bs+C) + Bs, \quad 2s = 2Bs + C.$$

The coefficient of s on the left is 2; on the right, it is $2B$. So $2B = 2$, and $B = 1$. Comparing constant terms on the left and right, I get $C = 0$.

Therefore,

$$\frac{1+2s^2}{s(s^2+1)} = \frac{1}{s} + \frac{s}{s^2+1}.$$

Using the transform table,

$$y = \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{s}{s^2+1}\right) = 1 + \cos t. \ \square$$

Example. As you can see, the computation of inverse transforms arises naturally in using transforms to solve differential equations. In this example, I'll focus on the process of computing an inverse transform. It is primarily an exercise in partial fractions.

I'll compute

$$\mathcal{L}^{-1}\left(\frac{s^3+10s^2+6s+20}{(s^2+4)(s+3)(s-2)}\right).$$

Set up the partial fractions decomposition:

$$\frac{s^3+10s^2+6s+20}{(s^2+4)(s+3)(s-2)} = \frac{As+B}{s^2+4} + \frac{C}{s+3} + \frac{D}{s-2},$$

$$s^3+10s^2+6s+20 = (As+B)(s+3)(s-2) + C(s^2+4)(s-2) + D(s^2+4)(s+3).$$

Setting $s = 2$ gives $80 = 40D$, so $D = 2$. Setting $s = -3$ yields $65 = -65C$, so $C = -1$. The equation becomes

$$s^3+10s^2+6s+20 = (As+B)(s+3)(s-2) - (s^2+4)(s-2) + 2(s^2+4)(s+3).$$

Setting $s = 0$ yields $20 = -6B + 8 + 24$, so $B = 2$. Hence,

$$s^3+10s^2+6s+20 = (As+2)(s+3)(s-2) - (s^2+4)(s-2) + 2(s^2+4)(s+3).$$

One approach here is to set s equal to anything other than 2, -3, or 0 and solve for A. For example, setting $s = 1$ yields $37 = -4(A+2) + 5 + 40$, so $A = 0$.

Another approach (which I don't recommend) is to multiply out the right side and equate coefficients of corresponding powers of s.

Anyway, I now have

$$\frac{s^3 + 10s^2 + 6s + 20}{(s^2+4)(s+3)(s-2)} = \frac{2}{s^2+4} - \frac{1}{s+3} + \frac{2}{s-2}.$$

Compute the inverse transform of each term on the right using the table:

$$\mathcal{L}^{-1}\left(\frac{s^3 + 10s^2 + 6s + 20}{(s^2+4)(s+3)(s-2)}\right) = \mathcal{L}^{-1}\left(\frac{2}{s^2+4} - \frac{1}{s+3} + \frac{2}{s-2}\right) = \sin 2t - e^{-3t} + 2e^{2t}. \quad \square$$

Example. $y'' - 2y' - 3y = te^t$, $y(0) = 1$, $y'(0) = 0$.

Take the transform of both sides:

$$\left(-0 - s + s^2 Y(s)\right) - 2\left(-1 + sY(s)\right) - 3Y(s) = \frac{1}{(s-1)^2}.$$

Simplify and solve for $Y(s)$:

$$Y(s) = \frac{s-2}{(s-3)(s+1)} + \frac{1}{(s-3)(s+1)(s-1)^2}.$$

This time, I'll use *Mathematica* to do the partial fractions decompositions:

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Apart[(s - 2)/((s - 3)(s + 1))]

     1            3
---------- + ---------
4 (-3 + s)   4 (1 + s)

Apart[1/((s - 3)(s + 1)(s - 1)^2)]

      1              1             1
----------- - ----------- - ----------
16 (-3 + s)              2   16 (1 + s)
              4 (-1 + s)
```

All together,

$$Y(s) = \frac{5}{16}\frac{1}{s-3} - \frac{11}{16}\frac{1}{s+1} - \frac{1}{4}\frac{1}{(s-1)^2}.$$

Therefore,

$$y(t) = \frac{5}{16}e^{3t} - \frac{11}{16}e^{-t} - \frac{1}{4}te^t. \quad \square$$

Example. (The **exponential shift**) Here is a formula which is often useful. If $F(s) = \mathcal{L}\left(f(t)\right)$,

$$\mathcal{L}\left(e^{ct} f(t)\right) = \int_0^\infty e^{ct} e^{-st} f(t)\, dt = \int_0^\infty e^{-(s-c)t} f(t)\, dt = F(s-c).$$

The transform formulas are

$$\mathcal{L}\left(e^{ct} f(t)\right) = F(s-c) \quad \text{and} \quad \mathcal{L}^{-1}\left(F(s-c)\right) = e^{ct} f(t).$$

The inverse transform formula says that if the given expression differs from that in the table by the replacement of s with $s-c$, you can take the inverse transform by using the table formula and tacking on an e^{ct}.

Here's an example. I'll compute

$$\mathcal{L}^{-1}\left(\frac{15}{s^2+4s+13}\right).$$

The right side of the table doesn't contain any quadratics in s with a middle term. However, the standard way of "removing" a middle term is to **complete the square**:

$$s^2+4s+13 = s^2+4s+4+9 = (s+2)^2+9,$$

$$\frac{15}{s^2+4s+13} = \frac{15}{(s+2)^2+9}.$$

The table contains an entry for $\dfrac{a}{s^2+a^2}$. This would match the expression above with $a=3$, if s were replaced with $s+2$. Apply the exponential shift with $c=-2$. First,

$$\mathcal{L}^{-1}\left(\frac{15}{s^2+9}\right) = 5\,\mathcal{L}^{-1}\left(\frac{3}{s^2+9}\right) = 5\sin 3t.$$

To complete the inverse transform, tack on an e^{-2t}:

$$\mathcal{L}^{-1}\left(\frac{15}{s^2+4s+13}\right) = 5e^{-2t}\sin 3t.\ \square$$

Example. $y''+4y'+4y = 8e^{-2t}$, $y(0)=1$, $y'(0)=1$.

Take the transform of both sides:

$$\mathcal{L}(y'')+2\,\mathcal{L}(y')+4\,\mathcal{L}(y) = 8\,\mathcal{L}\left(e^{-2t}\right),$$

$$\left(-y'(0)-sy(0)+s^2Y(s)\right)+4\left(-y(0)+sY(s)\right)+4Y(s) = \frac{8}{s+2}.$$

Simplify and solve for $Y(s)$:

$$Y(s) = \frac{8}{(s+2)^3}+\frac{s+5}{(s+2)^2}.$$

Partial fractions is unnecessary, since

$$\frac{s+5}{(s+2)^2} = \frac{s+2}{(s+2)^2}+\frac{3}{(s+2)^2}.$$

Therefore,

$$Y(s) = \frac{8}{(s+2)^3}+\frac{s+2}{(s+2)^2}+\frac{3}{(s+2)^2}.$$

The inverse transform is

$$y(t) = 4t^2e^{-2t}+3te^{-2t}+e^{-2t}.\ \square$$

Table of Laplace Transforms.

$f(t)$	$F(s) = \mathcal{L}(f(t))$
1	$\frac{1}{s}, \qquad s > 0$
e^{at}	$\frac{1}{s-a}, \qquad s > a$
$t^n, \qquad n \in \mathbf{Z}^+$	$\frac{n!}{s^{n+1}}, \qquad s > 0$
$t^r, \qquad r > 0$	$\frac{\Gamma(r+1)}{s^{r+1}}, \qquad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \qquad s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, \qquad s > 0$
$\sinh at$	$\frac{a}{s^2-a^2}, \qquad s > \lvert a\rvert$
$\cosh at$	$\frac{s}{s^2-a^2}, \qquad s > \lvert a\rvert$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \qquad s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \qquad s > a$
$t^n e^{at}, \qquad n \in \mathbf{Z}^+$	$\frac{n!}{(s-a)^{n+1}}, \qquad s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}, \qquad s > 0$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$f(ct)$	$\frac{1}{c}\,F\left(\frac{s}{c}\right), \qquad c > 0$
$f \star g = \int_0^t f(t-u)g(u)\,du$	$F(s)\cdot G(s)$
$\delta(t-c)g(t)$	$e^{-cs}g(c)$
$y^{(n)}(t)$	$-y^{(n-1)} - \ldots - s^{n-1}y(0) + s^nY(s)$
$(-t)^n f(t)$	$F^{(n)}(s)$