5-3-1994

Review Session for the Final - 2.

Example. Compute

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s-1)^2}e^{-2s}\right).$$

First, invert $\dfrac{1}{s^2(s-1)^2}$.

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)=t \quad \text{and} \quad \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right)=te^t.$$

Hence, the convolution formula gives

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s-1)^2}\right)=t\star te^t=\int_0^t (t-u)ue^u\,du=t\int_0^t ue^u\,du-\int_0^t u^2e^u\,du=$$

$$t\left[ue^u-e^u\right]_0^t-\left[u^2e^u-2ue^u+2e^u\right]_0^t=te^t-2e^t+t+2.$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s-1)^2}e^{-2s}\right)=u_2(t)\left((t-2)e^{t-2}-2(t-2)e^{t-2}+(t-2)+2\right).$$

For comparison, here's what you would go through to invert $\dfrac{1}{s^2(s-1)^2}$ without convolution. Set up the partial fractions decomposition:

$$\frac{1}{s^2(s-1)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}+\frac{D}{(s-1)^2}.$$

Clear denominators:

$$1=As(s-1)^2+B(s-1)^2+Cs^2(s-1)+Ds^2.$$

Setting $s=0$ yields $B=1$. Setting $s=1$ yields $D=1$. Plug these back in:

$$1=As(s-1)^2+(s-1)^2+Cs^2(s-1)+s^2.$$

Differentiate:

$$0=A(s-1)^2+2As(s-1)+2(s-1)+2Cs(s-1)+Cs^2+2s.$$

Setting $s=0$ yields $0=A-2$, $A=2$. Setting $s=1$ yields $0=C+2$, so $C=-2$. Therefore,

$$\frac{1}{s^2(s-1)^2}=\frac{2}{s}+\frac{1}{s^2}-\frac{2}{s-1}+\frac{1}{(s-1)^2}.$$

Hence,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s-1)^2}\right)=2+t-2e^t+te^t.$$

Now do the step function shift exactly as above. Obviously, convolution is much easier! □

Example. Solve

$$\left(3x^3(\ln x)^2 + y\right)\,dx - x\ln x\,dy = 0, \quad y(e) = e^3 + e.$$

Rewrite the equation:

$$\frac{dy}{dx} = \frac{3x^3(\ln x)^2 + y}{x\ln x} = 3x^2\ln x + \frac{1}{x\ln x}y, \quad \frac{dy}{dx} - \frac{1}{x\ln x}y = 3x^2\ln x.$$

The equation is first-order linear in y. The integrating factor is

$$I = \exp\int -\frac{1}{x\ln x}\,dx = \exp -(\ln(\ln x)) = \exp\ln\frac{1}{\ln x} = \frac{1}{\ln x}.$$

Therefore,

$$y\frac{1}{\ln x} = \int 3x^2\,dx = x^3 + C.$$

The solution is

$$y = x^3\ln x + C lnx.$$

When $x = e$, $y = e^3 + e$:

$$e^3 + e = e^3\ln e + C, \quad C = e.$$

Hence,

$$y = x^3\ln x + e lnx. \ \square$$

Example. Solve

$$(e^x + ye^x)\,dx - (1 + e^x)\,dy = 0, \quad y(0) = 3.$$

Separate variables:

$$e^x(1+y)\,dx = (1+e^x)\,dy, \quad \int\frac{e^x}{1+e^x}\,dx = \int\frac{dy}{1+y}, \quad \ln(1+e^x) + C = \ln|1+y|.$$

Exponentiate both sides and rename the arbitrary constant:

$$1 + y = C_0(1+e^x), \quad y = C_0(1+e^x) - 1.$$

When $x = 0$, $y = 3$:

$$3 = C_0(1+1) - 1, \quad C_0 = 2.$$

The solution is

$$y = 2(1+e^x) - 1. \ \square$$

Example. Solve the system

$$\vec{x}\,' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}\vec{x}.$$

The characteristic polynomial is $\lambda^2 + 2\lambda + 2$. The eigenvalues are $\lambda = -1 \pm i$.
For $\lambda = -1 + i$,

$$A - (-1+i)I = \begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix}.$$

An eigenvector $\langle a, b\rangle$ is a (nonzero) vector satisfying

$$\begin{bmatrix} 2-i & -5 \\ 1 & -2-i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In this situation, the two rows of $A - (-1+i)I$ must be multiples, or else there won't be any nontrivial solutions. So throw out the second row. The first row yields the equation

$$(2-i)a - 5b = 0.$$

By inspection, $a = 5$, $b = 2 - i$ works. Thus, $\langle 5, 2-i\rangle$ is an eigenvector.

Write down the solution for this eigenvector and find the real and imaginary parts:

$$e^{(-1+i)t} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} = e^{-t} \begin{bmatrix} 5\cos t & + & 5i\sin t \\ (2\cos t + \sin t) & + & i(-\cos t + 2\sin t) \end{bmatrix}.$$

The solution is

$$\vec{x} = c_1 e^{-t} \begin{bmatrix} 5\cos t \\ 2\cos t + \sin t \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 5\sin t \\ -\cos t + 2\sin t \end{bmatrix}.$$

You could do this using the matrix exponential, but *it is usually better to use the eigenvector method when you have complex eigenvalues.* □

Example. Find a solution to

$$y' + y = \begin{cases} x+1 & \text{if } x \le 1 \\ 2e^{-x} & \text{if } x > 1 \end{cases}, \qquad y(0) = 4$$

which is continuous for all x.

First, suppose $x \le 1$. The equation is

$$y' + y = x + 1.$$

It is first-order linear; the integrating factor is

$$I = \exp \int dx = e^x.$$

Thus,

$$ye^x = \int (x+1)e^x\,dx = xe^x + C, \quad y = x + Ce^{-x}.$$

Plug in the initial condition:

$$4 = y(0) = 0 + C, \quad C = 4.$$

Therefore,

$$y = x + 4e^{-x}.$$

Note that $y(1) = 1 + 4e^{-1}$.

Now suppose $x > 1$. The equation is

$$y' + y = 2e^{-x}.$$

It is first-order linear; the integrating factor is

$$I = \exp \int dx = e^x.$$

Therefore,

$$ye^x = \int 2\,dx = 2x + C', \quad y = 2xe^{-x} + C'e^{-x}.$$

I want to choose C' so this part of the solution fits together with the part I found on $x \le 1$ to give a continuous function. This means that the two pieces must match at $x = 1$. For the first part, $y(1) = 1+4e^{-1}$, so

$$1 + 4e^{-1} = y(1) = 2e^{-1} + C'e^{-1}, \quad C' = e + 2.$$

Therefore,

$$y = 2xe^{-x} + (e+2)e^{-x}.$$

The solution is

$$y = \begin{cases} x + 4e^{-x} & \text{if } x \le 1 \\ 2xe^{-x} + (e+2)e^{-x} & \text{if } x > 1 \end{cases}. \quad \square$$

Example. Use a Liapunov function of the form $H(x,y) = ax^2 + by^2$ to classify the critical point $(0,0)$ of the system

$$\frac{dx}{dt} = y - x^2y^2\sin x, \quad \frac{dy}{dt} = -x - xy^3\sin x.$$

If $H(x,y) = ax^2 + by^2$ and $a, b > 0$, then $H(0,0) = 0$ and $H(x,y) > 0$ for $(x,y) \ne (0,0)$. Hence, H is positive definite.

Next,

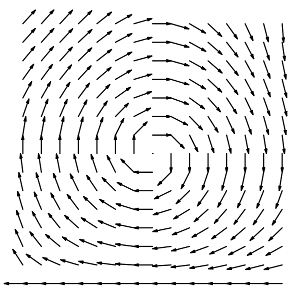
$$\nabla H \cdot \vec{x}\,' = \langle 2ax, 2by\rangle \cdot \langle y - x^2y^2\sin x, -x - xy^3\sin x\rangle = 2axy - 2ax^3y^2\sin x - 2bxy - 2bxy^4\sin x.$$

I can get the xy terms to cancel if I take $a = b$, so set $a = b = 1$. Then

$$\nabla H \cdot \vec{x}\,' = -2x^3y^2\sin x - 2xy^4\sin x = -2xy^2(\sin x)(x^2 + y^2).$$

For $(x,y) \ne (0,0)$, $x^2 + y^2 > 0$, $y^2 \ge 0$, and $x\sin x \ge 0$. (For the last inequality, notice that for x close to 0, $x > 0$ implies $\sin x > 0$, so $x\sin x > 0$, and $x < 0$ implies $\sin x < 0$, so $x\sin x > 0$.) Note that $\nabla H \cdot \vec{x}\,' = 0$ along the x and y axes, so I don't get strict inequality.

Hence, $-\nabla H \cdot \vec{x}\,' \ge 0$ for (x,y) near $(0,0)$, so $-\nabla H \cdot \vec{x}\,'$ is positive semidefinite. By case 1 of the stability theorem, $(0,0)$ is stable.



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