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Bernoulli equations and Riccati equations.

An equation of the form

$$y' + p(x)y = q(x)y^n,$$

where n is an integer, is a **Bernoulli equation**.

If $n = 0$, the equation is

$$y' + p(x)y = q(x).$$

The equation is first-order linear; you can solve it using an integrating factor:

$$I = \exp \int P\,dx.$$

If $n = 1$, the equation is

$$y' + p(x)y = q(x)y, \quad \text{or} \quad y' + (p(x) - q(x))\,y = 0.$$

You can solve this equation using separation of variables.

Suppose then that $n \neq 0, 1$. In this case, the substitution $v = y^{1-n}$ will reduce the equation to a first-order linear equation.

Example. $y' + 3y = e^{3x}y^2$

This is a Bernoulli equation with $n = 2$. Let $v = y^{-1}$, so $\dfrac{dv}{dx} = -\dfrac{1}{y^2}\dfrac{dy}{dx}$. Multiply by $-\dfrac{1}{y^2}$:

$$-\frac{1}{y^2}\frac{dy}{dx} - \frac{3}{y} = e^{3x}.$$

Substitute:

$$\frac{dv}{dx} - 3v = e^{3x}.$$

This equation is first-order linear in v. The integrating factor is

$$I = \exp \int (-3)\,dx = e^{-3x}.$$

Therefore,

$$ve^{-3x} = \int dx = x + C, \quad v = (x + C)e^{3x}.$$

Put y back:

$$\frac{1}{y} = (x + C)e^{3x}, \quad y = \frac{1}{(x + C)e^{3x}}. \square$$

A **Riccati equation** is an equation of the form

$$y' = a(x) + b(x)y + c(x)y^2.$$

Note that if $a(x) = 0$, this is a Bernoulli equation with $n = 2$.

Solving a Riccati equation depends on knowing a particular solution y_1. It may seem ridiculous to base a solution method on knowing a solution, but it's not that unreasonable — in many cases, you may be able to "guess" a solution, or you may notice one by inspection. This is the basis of a solution method called **reduction of order**.

Assume, then, that you know a particular solution y_1. "Particular" means that your solution may not be the most general, and will probably lack an arbitrary constant. The substitution $y = y_1 + \dfrac{1}{v}$ will reduce the Riccati equation to a first-order linear equation.

Example. Solve $e^{-x}y' = 1 - e^{2x} + 2e^x y - y^2$, using the fact that $y_1 = e^x$ is a solution.

You can verify that e^x is a solution by substitution:

$$e^{-x}y' = e^{-x} \cdot e^x = 1, \quad \text{while} \quad 1 - e^{2x} + 2e^x y - y^2 = 1 - e^{2x} + 2e^x \cdot e^x - e^{2x} = 1.$$

Let $y = e^x + \dfrac{1}{v}$, so $y' = e^x - \dfrac{1}{v^2}v'$. Substitute:

$$e^{-x}\left(e^x - \frac{1}{v^2}v'\right) = 1 - e^{2x} + 2e^x\left(e^x + \frac{1}{v}\right) - \left(e^x + \frac{1}{v}\right)^2.$$

After some simplification,

$$\frac{dv}{dx} = -e^x, \quad \text{so} \quad v = -e^x + C.$$

Put y back:

$$\frac{1}{y - e^x} = -e^x + C, \quad y = e^x + \frac{1}{-e^x + C}. \ \square$$

Example. Sometimes you can solve an equation by using a clever substitution. There's no method for being clever — the more experience you've had with calculus, the more likely it is that you'll see what to do.

For example, consider the equation

$$y' - 3x^2 = (x^3 - y + 2)^{1/3}.$$

When you study substitutions as a method of computing integrals, you're often told to substitute for something whose derivative is also there. The same idea applies here. Note that

$$\frac{d}{dx}\left(x^3 - y + 2\right) = 3x^2 - y'.$$

This is the negative of the left side of the equation. So let $u = x^3 - y + 2$. Then $\dfrac{du}{dx} = 3x^2 - y'$, so the equation becomes

$$\frac{du}{dx} = u^{1/3}.$$

Now the equation may be solved by separation of variables:

$$\int u^{-1/3}\,du = \int dx,$$

$$\frac{3}{2}u^{2/3} = x + c,$$

$$\frac{3}{2}\left(x^3 - y + 2\right)^{2/3} = x + c. \ \square$$

Example. (Tenenbaum and Pollard, *Ordinary Differential Equations*, page 101) Solve

$$(y - y^2 - x^2)\,dx - x\,dy = 0.$$

The equation is not separable, exact, homogeneous, or first order linear in x and y. It is not Bernoulli; it *is* Riccati, but a particular solution was not provided.

Here's the idea. Multiply everything out and rearrange the terms:

$$x\,dy - y\,dx = -(x^2 + y^2)\,dx.$$

The $x\,dy - y\,dx$ looks like the "top" of the Quotient Rule. It would be a Quotient Rule form if x^2 were on the bottom. So divide the equation by x^2:

$$\frac{x\,dy - y\,dx}{x^2} = -\left[1 + \left(\frac{y}{x}\right)^2\right]\,dx.$$

Now the left side is just $d\left(\dfrac{y}{x}\right)$. Conveniently, there's a $\dfrac{y}{x}$ on the right side as well. So let $u = \dfrac{y}{x}$. Then

$$du = -(1 + u^2)\,dx,$$

$$\int \frac{du}{1 + u^2} = -\int dx,$$

$$\tan^{-1} u = -x + c,$$

$$\tan^{-1} \frac{y}{x} = -x + c.$$

The idea in this problem was to *recognize an expression from a familiar formula* — namely, the Quotient Rule. □