Fourier series.

The solution to the one-dimensional heat equation with an arbitrary initial distribution is an infinite sum

$$\sum_{n=1}^{\infty} b_n \exp\left(-\frac{n^2\pi^2a^2t}{L^2}\right)\sin\left(\frac{\pi n x}{L}\right).$$

If $u(x,0)$ is the initial distribution, then

$$u(x,0)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{\pi n x}{L}\right).$$

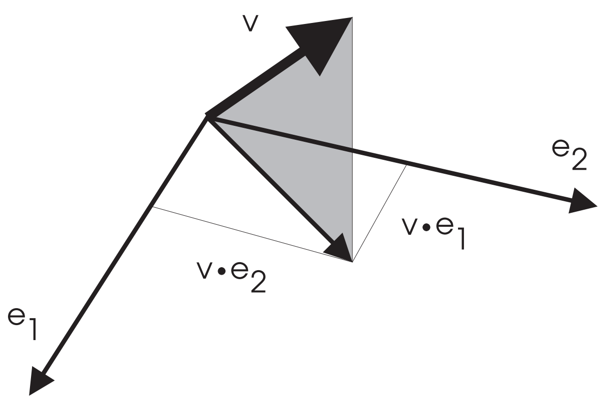
That is, $u(x,0)$ is expressed as an infinite sum of sines.

I'll consider the general problem of trying to represent a function $f(x)$ as an infinite sum of trigonometric functions:

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{\pi n x}{L}+b_n\sin\frac{\pi n x}{L}\right).$$

This sum is the **Fourier series** for $f(x)$.

The problem is analogous to one you've probably encountered in vector algebra. Suppose $\vec{v}$ is a vector in 3 dimensions and $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ are **orthonormal** — perpendicular vectors each of length 1. ($\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ are orthonormal vectors in $\mathbb{R}^3$.) $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ span a plane; the vector in this plane which is "closest" to $\vec{v}$ is the projection of $\vec{v}$ into the plane.



Moreover, the components of $\vec{v}$ in the $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ directions are the dot products

$$\vec{v}\cdot\overrightarrow{e_1}\ \text{and}\ \vec{v}\cdot\overrightarrow{e_2}.$$

Now here is the analogy. For the Fourier series of a function $f(x)$:

Instead of ...	**...you have ...**
A vector $\vec{v}$	A function $f(x)$
Orthonormal vectors $\overrightarrow{e_1}$, $\overrightarrow{e_2}$, ...	Orthonormal functions $\cos\dfrac{\pi n x}{L}$, $\sin\dfrac{\pi n x}{L}$ (for $n=0,1,\ldots$)
The dot product $\vec{a}\cdot\vec{b}$	$\langle f,g\rangle=\dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)g(x)\,dx$

The last line may seem puzzling. Think of the dot product of two vectors:

$$\langle 1, 2, 3\rangle \cdot \langle 4, 5, 6\rangle = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32.$$

It is a sum of products. Since an integral is like a sum, $\int_{-L}^{L} f(x)g(x)\,dx$ is in some sense a sum of products.

- $\int_{-L}^{L} f(x)g(x)\,dx$ is a dot product for functions.

Now functions are playing the role of vectors — in what sense are the sines and cosines like orthnormal vectors? This follows from the **orthogonality relations** for sine and cosine:

$$\frac{1}{L}\int_{-L}^{L} \cos\left(\frac{\pi jx}{L}\right)\cos\left(\frac{\pi kx}{L}\right)\,dx = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

$$\frac{1}{L}\int_{-L}^{L} \cos\left(\frac{\pi jx}{L}\right)\sin\left(\frac{\pi kx}{L}\right)\,dx = 0$$

$$\frac{1}{L}\int_{-L}^{L} \sin\left(\frac{\pi jx}{L}\right)\sin\left(\frac{\pi kx}{L}\right)\,dx = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}$$

Wow! This looks horrible, but all of these formulas are routine (if tedious) consequences of the angle addition formulas for sine and cosine.

Example. I'll illustrate the first formula with some numerical examples.

If I take different cosines, I should get 0. I'll try

$$\frac{1}{L}\int_{-L}^{L} \cos\left(\frac{3\pi x}{L}\right)\cos\left(\frac{7\pi x}{L}\right)\,dx.$$

Use the identity

$$\cos a \cos b = \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right).$$

I get

$$\cos\left(\frac{3\pi x}{L}\right)\cos\left(\frac{7\pi x}{L}\right) = \frac{1}{2}\left[\cos\left(\frac{10\pi x}{L}\right) + \cos\left(\frac{4\pi x}{L}\right)\right].$$

So

$$\frac{1}{L}\int_{-L}^{L} \cos\left(\frac{3\pi x}{L}\right)\cos\left(\frac{7\pi x}{L}\right)\,dx = \frac{1}{L}\int_{-L}^{L} \frac{1}{2}\left[\cos\left(\frac{10\pi x}{L}\right) + \cos\left(\frac{4\pi x}{L}\right)\right]\,dx =$$

$$\frac{1}{2L}\left[\frac{L}{10\pi}\sin\left(\frac{10\pi x}{L}\right) + \frac{L}{4\pi}\sin\left(\frac{4\pi x}{L}\right)\right]_{-L}^{L} = 0.$$

(Remember that the sine of a multiple of π is 0!)

If I do the integral with both cosines the same, I use the double angle formula:

$$\frac{1}{L}\int_{-L}^{L} \cos\left(\frac{3\pi x}{L}\right)\cos\left(\frac{3\pi x}{L}\right)\,dx = \frac{1}{2L}\int_{-L}^{L} \left[1 + \cos\left(\frac{6\pi x}{L}\right)\right]\,dx =$$

$$\frac{1}{2L}\left[x + \frac{L}{6\pi}\sin\left(\frac{6\pi x}{L}\right)\right]_{-L}^{L} = 1.$$

There is nothing essentially different in the derivation of the general formulas. □

To save writing, I'll write

$$\langle f, g\rangle \text{ for } \frac{1}{L}\int_{-L}^{L} f(x)g(x)\,dx.$$

Here are the orthogonality relations again:

$$\left\langle \cos\left(\frac{\pi jx}{L}\right), \cos\left(\frac{\pi kx}{L}\right)\right\rangle = \begin{cases} 0 & \text{if } j \ne k \\ 1 & \text{if } j = k \end{cases}$$
$$\left\langle \cos\left(\frac{\pi jx}{L}\right), \sin\left(\frac{\pi kx}{L}\right)\right\rangle = 0$$
$$\left\langle \sin\left(\frac{\pi jx}{L}\right) \sin\left(\frac{\pi kx}{L}\right)\right\rangle = \begin{cases} 0 & \text{if } j \ne k \\ 1 & \text{if } j = k \end{cases}$$

In words, this says that two different sines or cosines have dot product 0 — they're *perpendicular*. And if you dot a sine or a cosine with itself, you get 1. Remembering that $\vec{v}\cdot\vec{v} = |\vec{v}|^2$, this says that these sines and cosines have length 1.

(By the way, the formulas above work even if $j = 0$. $\sin 0 = 0$ isn't interesting, but $\cos 0 = 1$ means that I can throw the constant function 1 into our sine/cosine collection. That gives rise to the $\dfrac{a_0}{2}$ term in the Fourier series.)

Thus, *the formula*

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{\pi nx}{L} + b_n \sin\frac{\pi nx}{L}\right)$$

is simply an expression of f in terms of its components in the directions of the sines and cosines — thinking of functions as vectors!

Now recall that for "real" vectors, you can get the components of a vectors by dotting it with members of the orthonormal set. For Fourier series, I can get the components — the **Fourier coefficients** a_n, b_n — by dotting f with the sines and cosines. Here, then, are the formulas for the Fourier coefficients:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{\pi nx}{L}\right)\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{\pi nx}{L}\right)\,dx$$

The a_n formula correctly gives

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx.$$

However, you must compute this integral separately from the integral for the other a_n's.

It's possible to use the **complex exponential formula**

$$\exp\left(\frac{\pi ikx}{L}\right) = \cos\left(\frac{\pi kx}{L}\right) + i\sin\left(\frac{\pi kx}{L}\right)$$

to simplify computations.

Specifically, let

$$c_k = \frac{1}{L}\int_{-L}^{L} f(x)\exp\left(\frac{\pi kx}{L}\right)\,dx.$$

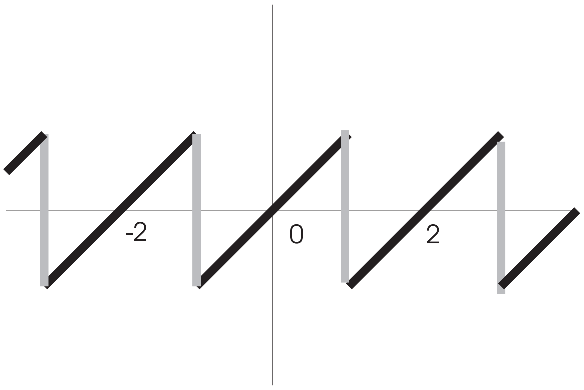
Then

$$a_k = \operatorname{re} c_k, \qquad b_k = \operatorname{im} c_k.$$

I will usually use this method to compute Fourier coefficients, except where symmetry considerations make it unnecessary. (I'll discuss **even** and **odd functions** later.)

Here's a final remark before I get to the examples. A sum of periodic functions is periodic. Since sine and cosine are periodic, you can only expect to represent periodic functions by a Fourier series. Therefore, outside the interval $-L \le x \le L$, you must "repeat" $f(x)$ in periodic fashion (so $f(x) = f(x + L)$ for all x).

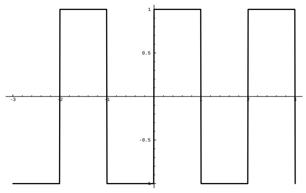
For example, consider the function $f(x) = x$, $-1 \le x < 1$. $L = 1$, so I must "repeat" the function every two units. Picture:



The Fourier expansion of $f(x)$ only converges to $y = x$ on the interval $-1 < x < 1$. Outside of that interval, it converges to the periodic function in the picture (except perhaps at the jump discontinuities, which I will discuss later).

Example. Find the Fourier expansion of

$$f(x) = \begin{cases} -1 & \text{if } -1 \le x < 0 \\ 1 & \text{if } 0 \le x < 1 \end{cases} \quad \text{and } f(x) = f(x+2) \text{ for all } x.$$



(Later on, I'll show that **odd functions** have $a_k = 0$ for all k. I'll proceed now as if I didn't know that.) Compute the constant term first:

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = 0.$$

Next, compute the c_n terms:

$$c_n = \frac{1}{1}\int_{-1}^{2} f(x)e^{\pi i n x}\,dx = -\int_{-1}^{0} e^{\pi i n x}\,dx + \int_{0}^{1} e^{\pi i n x}\,dx =$$

$$\left[\frac{i}{\pi n}e^{\pi i n x}\right]_{-1}^{0} + \left[-\frac{i}{\pi n}e^{\pi i n x}\right]_{0}^{1} = \frac{i}{\pi n}\left(1 - e^{-\pi i n}\right) - \frac{i}{\pi n}\left(e^{\pi i n} - 1\right).$$

Now

$$e^{-\pi n i} = \cos \pi n - i \sin \pi n = \cos \pi n = (-1)^n$$

$$e^{\pi n i} = \cos \pi n + i \sin \pi n = \cos \pi n = (-1)^n$$

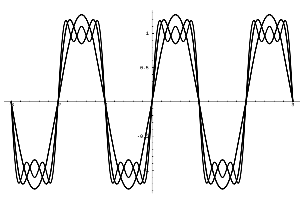
So

$$c_n = \frac{2i}{\pi n}(1 - (-1)^n).$$

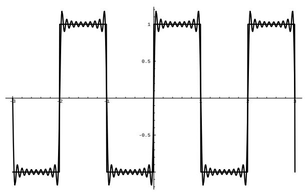
$a_n = \operatorname{re} c_n = 0$, while $b_n = \operatorname{im} c_n = \dfrac{2}{\pi n}(1 - (-1)^n)$. So the Fourier expansion is

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{\pi n}(1 - (-1)^n) \sin \pi n x.$$

Here is the graph of the first 5 partial sums of the series. You can see the successive sums converging to a square wave.



In the next graph, I've plotted the function together with the 20-th partial sum. They agree pretty closely. Notice the "ringing" at the points of discontinuity.



$\square$

Example. Find the Fourier expansion of

$$f(x) = \begin{cases} 1 & \text{if } -1 \le x < 0 \\ (x-1)^2 & \text{if } 0 \le x < 1 \end{cases} \quad \text{and } f(x) = f(x+2) \text{ for all } x.$$

Compute the constant term first:

$$c_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1\,dx + \int_{0}^{1} (x-1)^2\,dx = [x]_{-1}^{0} + \left[\frac{1}{3}(x-1)^3\right]_0^1 = \frac{4}{3}.$$

Next, compute the higher order terms:

$$c_n = \int_{-1}^{1} f(x) e^{\pi i n x}\,dx = \int_{-1}^{0} e^{\pi i n x}\,dx + \int_{0}^{1} (x-1)^2 e^{\pi i n x}\,dx =$$

$$\left[\frac{-i}{\pi n} e^{\pi i n x}\right]_{-1}^{0} + \left[\left(-\frac{i}{\pi n}(x-1)^2 + \frac{2}{\pi^2 n^2}(x-1) + \frac{2i}{\pi^3 n^3}\right) e^{\pi i n x}\right]_{0}^{1} =$$

$$-\frac{i}{\pi n}\left(1 - e^{-\pi n i}\right) + \frac{2i}{\pi^3 n^3} e^{\pi n i} + \frac{i}{\pi n} + \frac{2}{\pi^2 n^2} - \frac{2i}{\pi^3 n^3}.$$

Therefore,

$$c_n = -\frac{i}{\pi n}\left(1 - (-1)^n\right) + \frac{2i}{\pi^3 n^3}(-1)^n + \frac{i}{\pi n} + \frac{2}{\pi^2 n^2} - \frac{2i}{\pi^3 n^3} =$$

$$\frac{i}{\pi n}(-1)^n + \frac{2i}{\pi^3 n^3}(-1)^n + \frac{2}{\pi^2 n^2} - \frac{2i}{\pi^3 n^3}.$$

Now take real and imaginary parts:

$$a_n = \operatorname{re} c_n = \frac{2}{\pi^2 n^2}, \quad b_n = \operatorname{im} c_n = \frac{1}{\pi n}(-1)^n - \frac{2}{\pi^3 n^3}\left(1 - (-1)^n\right).$$

The Fourier series is

$$f(x) \sim \frac{2}{3} + \sum_{n=1}^{\infty} \left(\frac{2}{\pi^2 n^2} \cos \pi n x + \left(\frac{1}{\pi n}(-1)^n - \frac{2}{\pi^3 n^3}\left(1 - (-1)^n\right)\right) \sin \pi n x\right). \ \square$$