Interchanging the order of integration - double integrals.

An iterated integral

$$\int\!\!\int_D f(x,y)\,dx\,dy$$

can be computed by integrating with respect to x first or with respect to y first. In some cases, one order is better than the other. For this reason, we need to know how to go from a "bad" order of integration to a "good" order of integration.

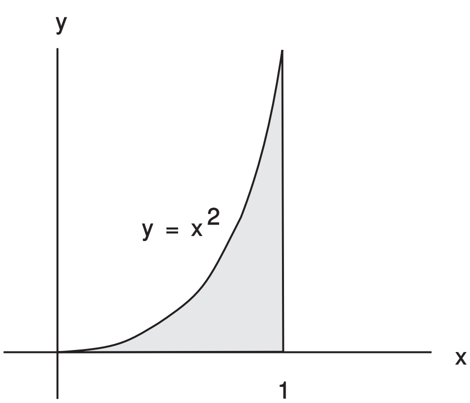
Example. $\displaystyle \int_0^1 \int_{\sqrt{y}}^1 \frac{1}{\sqrt{x^3+1}}\,dx\,dy$

As the integral is given, we'd need to integrate first with respect to x. However, we don't know the antiderivative of $\dfrac{1}{\sqrt{x^3+1}}$. We will interchange the order of integration and integrate first with respect to y.

Step 1: Pull off the limits of integration as inequalities.

$$0 \le y \le 1$$
$$\sqrt{y} \le x \le 1$$

Step 2: Draw the region defined by the inequalities.



Step 3: Describe the region by inequalities with the variables in the opposite order.

In the first set of inequalities, y came first. In this set, x will come first. For x, we simply take the numerical bounds in the x-direction: $0 \le x \le 1$.

Next, we need the inequalities for y. y is the *vertical* variable, so it will be bounded by expressions for the *bottom curve* and the *top curve* of the region. The bottom curve is the x-axis, which is $y = 0$. The top curve is $x = \sqrt{y}$. We express y in terms of x: $y = x^2$.

Therefore, the inequalities for y are $0 \le y \le x^2$. Our new set of inequalities is

$$0 \le x \le 1$$
$$0 \le y \le x^2$$

Step 4: Put the inequalities back onto the integral:

$$\int_0^1 \int_0^{x^2} \frac{1}{\sqrt{x^3+1}}\, dy\, dx = \int_0^1 \frac{1}{\sqrt{x^3+1}} \int_0^{x^2} dy\, dx =$$
$$\int_0^1 \frac{1}{\sqrt{x^3+1}} \left[y\right]_0^{x^2} dx = \int_0^1 \frac{x^2}{\sqrt{x^3+1}}\, dx =$$
$$\left[\frac{2}{3}\sqrt{x^3+1}\right]_0^1 = \frac{2}{3}(\sqrt{2}-1) \approx 0.27614. \square$$

Schematically, here is what we're doing:

$$\begin{array}{c} \text{integral} \\ \downarrow \\ \text{inequalities} \\ \downarrow \\ \text{picture} \\ \downarrow \\ \text{inequalities} \\ \downarrow \\ \text{integral} \end{array}$$

We will use essentially the same procedure to **convert an integral from rectangular to polar**.

Problems on Interchanging the Order of Integration.

1. Compute

$$\int_1^e \int_{\ln y}^1 \cos(e^x - x)\, dx\, dy$$

by interchanging the order of integration.

2. Express

$$\int_0^1 \int_0^y f(x,y)\, dx\, dy + \int_1^2 \int_0^{2-y} f(x,y)\, dx\, dy$$

as a single iterated integral by interchanging the order of integration.

3. Compute

$$\int_0^1 \int_y^1 2\cos(x^2)\, dx\, dy.$$

Solutions to the Problems on Interchanging the Order of Integration.

1. Compute

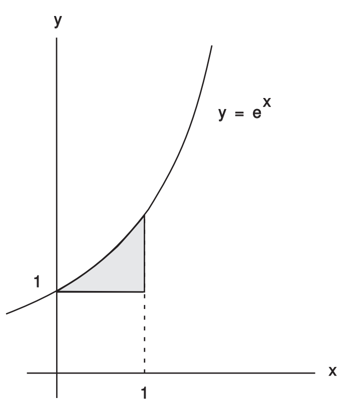
$$\int_1^e \int_{\ln y}^1 \cos(e^x - x)\, dx\, dy$$

by interchanging the order of integration.

Pull off the limits as inequalities:

$$1 \le y \le e$$
$$\ln y \le x \le 1$$

Next, draw the region determined by the inequalities. The inequalities $1 \le y \le e$ tell us that the region lies in the horizontal strip between $y = 1$ (bottom) and $y = e$ (top).



The inequalities $\ln y \le x \le 1$ give us the *left-hand* and *right-hand* boundaries, because x is the horizontal variable. The left-hand curve is $x = \ln y$, or $y = e^x$. The right-hand curve is $x = 1$. The region is shown above.

Next, describe the region by inequalities with the variables switched. We do x first. The numerical bounds on x are 0 and 1, so $0 \le x \le 1$.

To get the bounds on y, we look at the *bottom curve* and the *top curve*. The bottom curve is the line $y = 1$. The top curve is $y = e^x$. Hence, we have $1 \le y \le e^x$.

Our new inequalities are

$$0 \le x \le 1$$
$$1 \le y \le e^x$$

Put the inequalities back onto the integral:

$$\int_0^1 \int_1^{e^x} \cos(e^x - x)\, dy\, dx = \int_0^1 \cos(e^x - x) \int_1^{e^x} dy\, dx =$$
$$\int_0^1 \cos(e^x - x)\, [y]_1^{e^x}\, dx = \int_0^1 (e^x - 1)\cos(e^x - x)\, dx =$$
$$[u = e^x - x \quad du = (e^x - 1)\, dx]$$
$$[\sin(e^x - x)]_0^1 = \sin(e - 1) - \sin 1 \approx 0.14767. \square$$

2. Express

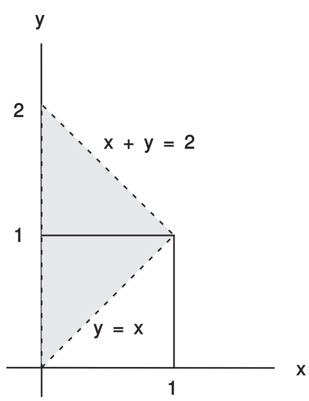
$$\int_0^1 \int_0^y f(x, y)\, dx\, dy + \int_1^2 \int_0^{2-y} f(x, y)\, dx\, dy$$

as a single iterated integral by interchanging the order of integration.

Pull off the limits as inequalities:

$$\begin{array}{cc} 0 \le y \le 1 & 1 \le y \le 2 \\ 0 \le x \le y & 0 \le x \le 2 - y \end{array}$$

Draw the region determined by the inequalities.



Describe the region by inequalities with the variables switched:

$$\begin{array}{c} 0 \le x \le 1 \\ x \le y \le 2 - x \end{array}$$

Put the new inequalities back onto the integral:

$$\int_0^1 \int_x^{2-x} f(x, y)\, dy\, dx. \square$$

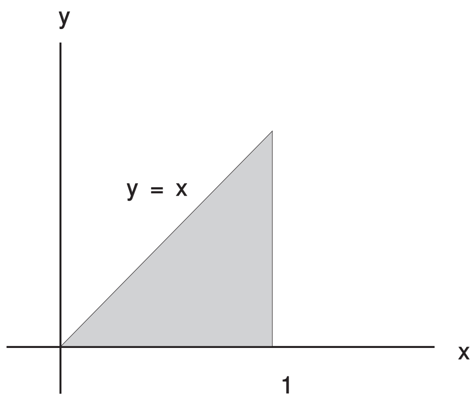
3. Compute

$$\int_0^1 \int_y^1 2\cos(x^2)\, dx\, dy.$$

Pull off the limits as inequalities:

$$\begin{array}{c} 0 \le y \le 1 \\ y \le x \le 1 \end{array}$$

Draw the region determined by the inequalities.



Describe the region by inequalities with the variables switched:

$$\begin{array}{c} 0 \le x \le 1 \\ 0 \le y \le x \end{array}$$

Put the new inequalities back onto the integral:

$$\int_0^1 \int_y^1 2\cos(x^2)\, dx\, dy = \int_0^1 \int_0^x 2\cos(x^2)\, dy dx = \int_0^1 2x\cos(x^2)\, dx =$$

$$\left[\sin(x^2)\right]_0^1 = \sin 1 \approx 0.84147. \ \square$$