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Elementary Solution Methods for Differential Equations.

This is a review of four elementary solution methods for differential equations. **First-order linear equations** are equations which can be written in the form

$$y' + P(x)y = Q(x).$$

You can solve a first-order linear equation using the **integrating factor**

$$I = \exp \int P(x)\,dx.$$

An equation in x and y is **separable** if you can move all the x's to one side and all the y's to the other. Then solving the equation is simply a matter of integrating both sides.

An equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is **exact** if $M = \dfrac{\partial f}{\partial x}$ and $N = \dfrac{\partial f}{\partial y}$ for some function $f(x, y)$. If you can find the potential function f, the equation becomes $df = 0$; integrating gives $f = c$, and the equation is solved.

An equation

$$\frac{dy}{dx} = h(x, y)$$

is **homogeneous** if $h(x, y)$ can be written entirely in terms of $\dfrac{y}{x}$. In that case, the substitution $v = \dfrac{y}{x}$ leads to an equation which can be solved by separation of variables.

That is a brief overview; now I'll look at the solution methods in more detail.

1. First-order linear equations.

A **first-order linear equation** is an equation of the form

$$y' + P(x)y = Q(x).$$

You can solve this equation by multiplying the equation by an **integrating factor**. After multiplication, the left side will be exactly the derivative of a function, and you can solve the equation by integrating both sides.

Multiply the equation by $I = \exp \int P(x)\,dx$. Then

$$Iy' + IPy = IQ.$$

By the Product Rule and the Chain Rule,

$$\frac{d}{dx}(yI) = Iy' + I'y = Iy' + \left(\frac{d}{dx}\exp \int P(x)\,dx\right) y = Iy' + \left(\exp \int P(x)\,dx\right) \cdot P \cdot y = Iy' + IPy.$$

The last expression is the same as the left side of $Iy' + IPy = IQ$. Hence,

$$\frac{d}{dx}(yI) = IQ.$$

Integrate both sides:

$$yI = \int IQ\,dx.$$

Since I and Q are both known, you only need to perform the integration in order to complete the solution.

To summarize, you can solve a first-order linear equation $y' + P(x)y = Q(x)$ as follows:

1. Compute the integrating factor $I = \exp \int P(x)\, dx$.

2. Write down the equation $yI = \int IQ\, dx$ and finish by performing the integration.

Example. $(x^3 \cos x + 2y)\, dx - x\, dy = 0$.

Rewrite the equation as

$$\frac{dy}{dx} - \frac{2}{x} y = x^2 \cos x.$$

The equation is a first-order linear equation with $P = -\frac{2}{x}$ and $Q = x^2 \cos x$. The integrating factor is

$$I = \exp \int -\frac{2}{x}\, dx = \exp(-2 \ln x) = \exp(\ln x^{-2}) = x^{-2} = \frac{1}{x^2}.$$

(You do not need to add an arbitrary constant when you compute an integrating factor.)

Therefore,

$$\frac{1}{x^2} \cdot y = \int \frac{1}{x^2} x^2 \cos x\, dx = \int \cos x\, dx = \sin x + C.$$

The solution is

$$y = x^2 \sin x + C. \square$$

Example. $\dfrac{dy}{dx} = \dfrac{y^2}{e^y - 3xy}$.

The equation is not a linear equation in y. However, solving for $\dfrac{dx}{dy}$ produces

$$\frac{dx}{dy} = \frac{e^y - 3xy}{y^2} = \frac{1}{y^2} e^y - \frac{3}{y} x.$$

This is a linear equation in x. Rewrite it as

$$\frac{dx}{dy} + \frac{3}{y} x = \frac{1}{y^2} e^y.$$

The integrating factor is

$$I = \exp \int \frac{3}{y}\, dy = \exp(3 \ln y) = \exp(\ln y^3) = y^3.$$

Hence,

$$xy^3 = \int ye^y\, dy = ye^y - e^y + C.$$

The solution is

$$x = \frac{1}{y^3}(ye^y - e^y + C). \square$$

Example. A first-order linear equation $y' + P(x)y = Q(x)$ is guaranteed to have a unique solution satisfying the initial condition $y(x_0) = y_0$ on any interval containing x_0 on which P and Q are continuous.

For example, consider the initial value problem

$$y' + \frac{y}{x^2 - 4} = \frac{x^2 + 1}{x}, \quad y(1) = 4.$$

$\dfrac{y}{x^2 - 4}$ and $\dfrac{x^2 + 1}{x}$ are continuous on the intervals $x < -2$, $-2 < x < 0$, $0 < x < 2$, and $x > 2$.

Even without solving the equation, you are guaranteed a unique solution to the initial value problem. Since the initial condition is $x = 1$, $y = 4$, and since $x = 1$ is in the interval $0 < x < 2$, the solution will be valid on the interval $0 < x < 2$. □

2. Separation of variables.

In some cases, it's possible to rearrange a first-order equation $\dfrac{dy}{dx} = f(x, y)$ by "moving all the x's to one side and all the y's to the other". This is called **separation of variables**. If the equation is separable, you can solve the separated equation by integrating both sides.

Example. Solve the equation $y' \sin x - y \cos x = 0$, $y\left(\dfrac{\pi}{2}\right) = 1$. What does the existence and uniqueness theorem have to say about the validity of the solution?

Move the x's to one side, the y's to the other, then integrate both sides:

$$\frac{dy}{dx}\sin x - y\cos x = 0$$

$$\frac{dy}{dx}\sin x = y\cos x$$

$$\int \frac{1}{y}\,dy = \int \frac{\cos x}{\sin x}\,dx$$

$$\ln|y| = \ln|\sin x| + C$$

Note that you only write *one* arbitrary constant. One way to think about this is that you've performed a *single* integration step (on both sides at once).

From another point of view, suppose you'd written

$$\ln|y| + C_1 = \ln|\sin x| + C_2.$$

Then

$$\ln|y| = \ln|\sin x| + C_2 - C_1.$$

Let $C = C_2 - C_1$ (they are all arbitrary constants!). Then

$$\ln|y| = \ln|\sin x| + C.$$

This is the same as before.

Continuing with the problem, insert the initial condition to solve for C. When $x = \dfrac{\pi}{2}$, $y = 1$:

$$\ln 1 = \ln 1 + C, \quad \text{so} \quad C = 0.$$

The solution is

$$\ln|y| = \ln|\sin x|.$$

Exponentiating both sides gives $|y| = |\sin x|$, or $y = \pm\sin x$. However, since $y = 1$ (positive) when $x = \dfrac{\pi}{2}$, the solution is $y = \sin x$.

The original equation can be written in the form

$$\frac{dy}{dx} - y\,\frac{\cos x}{\sin x} = 0.$$

It is a linear equation, and $\dfrac{\cos x}{\sin x}$ is continuous for $x \ne n\pi$, where n is an integer. The largest interval which contains $x = \dfrac{\pi}{2}$ and does not contain a point of the form $x = n\pi$ is $0 < x < \pi$. Therefore, the existence and uniqueness theorem guarantees the validity of the solution on the interval $0 < x < \pi$. □

Example. In some cases, you may have to stare at the equation for a minute in order to see how to separate it. Consider the equation

$$\frac{dy}{dx} = e^{x-y}.$$

In this case, you need to remember the rules for exponents:

$$e^{x-y} = e^x \cdot e^{-y}.$$

Now the problem is easy:

$$\frac{dy}{dx} = e^x \cdot e^{-y}$$

$$\int e^y\,dy = e^x\,dx$$

$$e^y = e^x + C$$

$$y = \ln\left(e^x + C\right). \ \square$$

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3. Exact equations and integrating factors.

A differential equation

$$M\,dx + N\,dy = 0$$

is **exact** if

$$M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y}$$

for some function $f(x, y)$.

The terminology makes sense, because in this situation the equation is

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0, \quad \text{or} \quad df = 0.$$

That is, the left is is *exactly* df, for some function f.

If you can find f, it is easy to solve the equation: Just integrate both sides:

$$df = 0 \quad \text{gives} \quad f = C.$$

There are two questions I need to answer. First, how can you tell whether a given equation is exact? Second, if an equation is exact, how do you find the function f?

The first question is easy to answer, because it is really the same as a question in multivariable calculus. To say that

$$M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y}$$

is exact is the same as saying that the **vector field** $\langle M, N\rangle$ is a **gradient field** (i.e. a **conservative field**). The function f is a **potential function** for the vector field. From multivariable calculus, $\langle M, N\rangle$ is a gradient field if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

That is, to determine whether the equation $M\,dx + N\,dy = 0$ is exact, check whether $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Assuming that an equation is exact, how do you find a potential function f? There is an easy method for doing this which you probably learned in multivariable calculus; I'll review it in the examples below.

Example. $(3x^2y + y^3)\,dx + (x^3 + 3xy^2)\,dy = 0$, $y(2) = 1$.

$$\frac{\partial M}{\partial y} = 3x^2 + 3y^2 \quad \text{and} \quad \frac{\partial N}{\partial x} = 3x^2 + 3y^2.$$

Since $M_y = N_x$, the equation is exact.

I want to find a function $f(x, y)$ such that

$$\frac{\partial f}{\partial x} = M = 3x^2y + y^3, \quad \frac{\partial f}{\partial y} = N = x^3 + 3xy^2.$$

Integrate the first equation with respect to x (holding y constant):

$$f = \int (3x^2y + y^3)\,dx = x^3y + xy^3 + h(y).$$

$h(y)$ is the arbitrary constant; it is a function of y, because y was held constant.

Next, differentiate the last equation with respect to y and set the result equal to N:

$$x^3 + 3xy^2 = N = \frac{\partial f}{\partial y} = x^3 + 3xy^2 + \frac{dh}{dy}.$$

Comparing terms gives $\dfrac{dh}{dy} = 0$.

If you integrate this equation with respect to y, you find that h is a *numerical* constant. In this situation, just take $h = 0$; any other constant value doesn't change the derivatives of f, and all you want is *some* potential function. Now

$$f = x^3y + xy^3.$$

The original differential equation says that $df = 0$, which means that $f = C$. The solution is

$$x^3y + xy^3 = C.$$

Finally, insert the initial condition to solve for C. $y(2) = 1$ means that when $x = 2$, $y = 1$. Plug these values into the general solution:

$$8 \cdot 1 + 2 \cdot 1 = C, \quad \text{so} \quad C = 10.$$

The solution is

$$x^3 y + x y^3 = 10. \square$$

Remarks:

- It is very easy to stop after finding f, but the *solution* is $f = C$. Remember that the general solution to a first-order equation *should* have an arbitrary constant somewhere!
- As you saw in this example, you *don't* need to insert an arbitrary constant when you integrate to find $h(y)$.

In some cases, the equation may not be exact as given, but will become exact when multiplied by an **integrating factor**.

Example. The equation

$$\left(\frac{x^2}{y} + y + 2x^2\right) dx - \left(\frac{x^3}{y^2} + x\right) dy = 0$$

is not exact, because

$$M_y = -\frac{x^2}{y^2} + 1, \quad N_x = -\frac{3x^2}{y^2} - 1$$

and $M_y \neq N_x$.

Multiply by the equation by $\dfrac{1}{x^2}$:

$$\left(\frac{1}{y} + \frac{y}{x^2} + 2\right) dx - \left(\frac{x}{y^2} + \frac{1}{x}\right) dy = 0.$$

For the new equation,

$$M_y = -\frac{1}{y^2} + \frac{1}{x^2}, \quad N_x = -\frac{1}{y^2} + \frac{1}{x^2}.$$

The new equation is exact. $\square$

An expression I is an **integrating factor** for $M\,dx + N\,dy = 0$ if

$$IM\,dx + IN\,dy = 0$$

is exact.

To be exact means that

$$(IM)_y = (IN)_x,$$

or

$$N\frac{dI}{dx} - M\frac{dI}{dy} = \left(\frac{dM}{dy} - \frac{dN}{dx}\right) I.$$

You can't expect to solve this partial differential equation for I without more information. Here are two important cases:

- If $\dfrac{M_y - N_x}{N}$ is a function of x alone, then an integrating factor is given by

$$I = \exp \int \frac{M_y - N_x}{N}\, dx.$$

- If $\dfrac{N_x - M_y}{M}$ is a function of y alone, then an integrating factor is given by

$$I = \exp \int \frac{N_x - M_y}{M}\, dy.$$

Example. The first-order linear equation

$$y' + P(x)y = Q(x)$$

can be rewritten as

$$\left(P(x)y - Q(x)\right)\, dx + \, dy = 0.$$

Now

$$M_y = P(x), \quad N_x = 0,$$

so $\dfrac{M_y - N_x}{N} = P(x)$, a function of x alone.

The integrating factor is

$$I = \exp \int P(x)\, dx.$$

You probably recognize this as the integrating factor you use to solve a first-order linear equation. □

Example. $2xy \ln y\, dx + (x^2 + y^2\sqrt{y^2 + 1})\, dy = 0.$

$$M_y = 2x \ln y + 2x \quad \text{but} \qquad N_x = 2x.$$

The equation is not exact. However,

$$\frac{N_x - M_y}{M} = -\frac{1}{y}.$$

This is a function of y alone. Therefore, an integrating factor is given by

$$I = \exp \int -\frac{1}{y}\, dy = \exp - \ln y = \frac{1}{y}.$$

Multiply the original equation by $\dfrac{1}{y}$:

$$2x \ln y\, dx + \left(\frac{x^2}{y} + y\sqrt{y^2 + 1}\right)\, dy = 0.$$

Then

$$M_y = \frac{2x}{y} \quad \text{and} \quad N_x = \frac{2x}{y}.$$

The new equation is exact.

I want to find a function f such that $\dfrac{\partial f}{\partial x} = 2x \ln y$ and $\dfrac{\partial f}{\partial y} = \dfrac{x^2}{y} + y\sqrt{y^2 + 1}$. Integrate M with respect to x:

$$f = \int 2x \ln y\, dx = x^2 \ln y + h(y).$$

Differentiate with respect to y:

$$\frac{x^2}{y} + y\sqrt{y^2+1} = \frac{\partial f}{\partial y} = \frac{x^2}{y} + \frac{dh}{dy}.$$

Hence, $\frac{dh}{dy} = y\sqrt{y^2+1}$, and $h = \frac{1}{3}(y^2+1)^{3/2}$. Thus, $f = x^2 \ln y + \frac{1}{3}(y^2+1)^{3/2}$.
The solution is

$$x^2 \ln y + \frac{1}{3}(y^2+1)^{3/2} = C. \square$$

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4. Homogeneous equations.

A function $f(x, y)$ is **homgeneous of degree n** if

$$f(ax, ay) = a^n f(x, y).$$

This says, roughly, that all the terms in f occur to the power n.

Examples. $x^2 + 3xy - y^2$ is homogeneous of degree 2:

$$(ax)^2 + 3(ax)(ay) - (ay)^2 = a^2(x^2 + 3xy - y^2).$$

$\ln y - \ln x$ is homogeneous of degree 0:

$$\ln ay - \ln ax = \ln \frac{ay}{ax} = \ln \frac{y}{x} = a^0(\ln y - \ln x).$$

Likewise, $\cos 2x$ is not homogeneous (the power of x is trapped inside the cosine, which is like an undefined power). However, $\cos \frac{2y}{x}$ is homogeneous of degree 0.

$x^2 - 5y^2 + 1$ is not homogeneous: $(ax)^2 - 5(ay)^2 + 1$ cannot be simplified to $a^n(x^2 - 5y^2 + 1)$ for any n.

$\frac{x^3 - x^2y}{3xy^2 + 7y^3}$ is homogeneous of degree 0:

$$\frac{(ax)^3 - (ax)^2(ay)}{3(ax)(ay)^2 + 7(ay)^3} = \frac{a^3(x^3 - x^2y)}{a^3(3xy^2 + 7y^3)} = \frac{x^3 - x^2y}{3xy^2 + 7y^3} = a^0 \frac{x^3 - x^2y}{3xy^2 + 7y^3}.$$

The last example illustrates an important point. The numerator $x^3 - x^2y$ is homogeneous of degree 3; the denominator $3xy^2 + 7y^3$ is homogeneous of degree 3. The result was homogeneous of degree $3 - 3 = 0$. $\square$

Fact 1: If M and N are homogeneous of the same degree, then $\frac{M}{N}$ is homogeneous of degree 0.

Here is something else you can see from the last example:

$$\frac{x^3 - x^2y}{3xy^2 + 7y^3} = \frac{\frac{1}{x^3}(x^3 - x^2y)}{\frac{1}{x^3}(3xy^2 + 7y^3)} = \frac{1 - \frac{y}{x}}{3\left(\frac{y}{x}\right)^2 + 7\left(\frac{y}{x}\right)^3}.$$

After some algebra, the fraction was expressed *entirely* in terms of $\frac{y}{x}$. This works in general:

Fact 2: If $f(x, y)$ is homogeneous of degree 0, then f can be written entirely in terms of $\dfrac{y}{x}$.

A differential equation

$$M\,dx + N\,dy = 0$$

is **homogeneous** if M and N are homogeneous functions *of the same degree*. The key point is that *a homogeneous equation can be solved using the two facts above.*

Write the equation in the form

$$\frac{dy}{dx} = -\frac{M}{N}.$$

The right side is the quotient of two homogeneous forms of the same degree, so it is homogeneous of degree 0 (Fact 1). Therefore, it can be written entirely in terms of $\dfrac{y}{x}$ (Fact 2). Hence, you can make the substitution

$$v = \frac{y}{x}, \quad \text{or} \quad y = vx, \quad \text{so} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

This will reduce the equation to one that can be integrated using separation of variables.

Summary:

- If you think an equation is homogeneous, write it in the form $\dfrac{dy}{dx} = f(x, y)$. If you can express $f(x, y)$ entirely in terms of $\dfrac{y}{x}$'s, the equation is homogeneous and this substitution will work.

Example. $x\dfrac{dy}{dx} = y(\ln y - \ln x)$.

Rewrite the equation as

$$\frac{dy}{dx} = \frac{y}{x}\ln\frac{y}{x}.$$

The right side has been written entirely in terms of $\dfrac{y}{x}$: The equation is homogeneous. Let

$$v = \frac{y}{x}, \quad \text{so} \quad y = vx \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Then

$$v + x\frac{dv}{dx} = v\ln v$$

$$x\frac{dv}{dx} = v\ln v - v = v(\ln v - 1)$$

Now separate variables. *If you're doing things correctly, this method must lead to a separable equation.*

$$\int \frac{1}{v(\ln v - 1)}\,dv = \int \frac{dx}{x}$$

$$\ln|\ln v - 1| = \ln|x| + C$$

Normally, simplification is optional, but I will show you how to do this since it is sometimes useful. Exponentiate both sides:

$$|\ln v - 1| = e^{\ln|x|}e^{C} = e^{C}|x|.$$

Think of the $|\cdot|$'s as being $\pm$. Remove the $|\cdot|$'s, grouping the $\pm$ with e^C — then rename $\pm e^C$ to C_0:

$$\ln v - 1 = C_0 x.$$

Finally, you should always put the original variable back:

$$\ln \frac{y}{x} - 1 = C_0 x. \square$$

Example. $(4x - 3y)\,dx + (2y - 3x)\,dy = 0$.

Rewrite the equation as

$$\frac{dy}{dx} = -\frac{4x - 3y}{2y - 3x} = -\frac{4 - 3\dfrac{y}{x}}{2\dfrac{y}{x} - 3}.$$

The right side has been written entirely in terms of $\dfrac{y}{x}$: The equation is homogeneous. Let

$$v = \frac{y}{x}, \quad \text{so} \quad y = vx \quad \text{and} \quad \frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Substitute and separate variables:

$$\begin{aligned}
v + x\frac{dv}{dx} &= -\frac{4 - 3v}{2v - 3} \\
x\frac{dv}{dx} = -\frac{4 - 3v}{2v - 3} - v &= \frac{-2(v^2 - 3v + 2)}{2v - 3} \\
\int \frac{2v - 3}{v^2 - 3v + 2},\, dv &= -2\int \frac{dx}{x} \\
\int \left(\frac{1}{v - 1} + \frac{1}{v - 2}\right)\, dv &= -2\int \frac{dx}{x} \\
\ln|v - 1| + \ln|v - 2| &= -2\ln|x| + C \\
\ln|(v - 1)(v - 2)| &= \ln\frac{1}{x^2} + C \\
(v - 1)(v - 2) &= \frac{C_0}{x^2} \\
\left(\frac{y}{x} - 1\right)\left(\frac{y}{x} - 2\right) &= \frac{C_0}{x^2} \\
(y - x)(y - 2x) &= C_0
\end{aligned}$$

Here's the work for the partial fractions:

$$\frac{2v - 3}{v^2 - 3v + 2} = \frac{2v - 3}{(v - 1)(v - 2)} == \frac{A}{v - 1} + \frac{B}{v - 2}$$

$$2v - 3 = A(v - 2) + B(v - 1)$$

Setting $v = 1$ gives $-1 = -A$, so $A = 1$. Setting $v = 2$ gives $1 = B$. Therefore,

$$\frac{2v - 3}{v^2 - 3v + 2} = \frac{1}{v - 1} + \frac{1}{v - 2}. \square$$

Problems on Elementary Solution Methods.

1. Solve: $(1+y^2)\,dx - x\,dy = 0$.

2. Solve: $y' = \dfrac{2}{x}y + \dfrac{x^2}{\sqrt{1+x^2}}$.

3. Solve: $(y\cos x - \cos x)\,dx + dy = 0$, $y(0) = 1$.

4. Solve: $\dfrac{dy}{dx} = \dfrac{y}{2y\ln y + y - x}$, $y(2) = 1$.

5. Solve: $y' = \sin(x-y)$.

6. Solve the equation $\dfrac{dy}{dx} = x^3(1-y)$, $y(0) = 3$. For what range of x values is the solution valid?

7. Solve: $e^{-y}(1+y') = 1$.

8. Solve: $(3x^2 - y^2)\,dx + \left(xy - \dfrac{x^3}{y}\right)\,dy = 0$.

9. A 40-gallon tank is filled with pure water. Water containing 1 pound of dissolved salt per gallon enters at 4 gallons per minutes; the well-stirred mixture drains out at 8 gallons per minute. How much salt is in the tank when it is half-empty?

10. On what interval does the initial value problem

$$\frac{dy}{dx} + \frac{\sin x}{x}y = \frac{1}{x+5}, \quad y(-6) = 4$$

have a unique solution?

11. Solve: $(3x^2 - 2x - y)\,dx + (2y - x + 3y^2)\,dy = 0$.

12. (a) The equation

$$(2y - 5x - 7)\,dx + (2x - y + 3)\,dy = 0$$

is not homogeneous, because (e.g.) $-5x + 2y - 7$ is not homogeneous. However, it becomes homogeneous if you let

$$x = u + h, \quad y = v + k.$$

Perform this substitution and find values for h and k which make the constant terms in the coefficients of dx and dy vanish. Then solve the resulting equation.

(b) The equation

$$(8x + 4y + 1)\,dx + (4x + 2y + 1)\,dy = 0$$

appears to be similar to the equation in part (a), but the substitution used in part (a) won't work: You will find it impossible to find values for h and k which make the constant terms vanish.

The substitution fails because

$$8x + 4y = 4(2x + y) \quad \text{while} \quad 4x + 2y = 2(2x + y).$$

Take advantage of this fact by letting $z = 2x + y$. (Don't forget to substitute for the differentials!) Then solve the resulting equation.

(c) Can you solve $(8x+4y+1)\,dx + (4x+2y+1)\,dy = 0$ without using the method of homogeneous equations?

13. Solve: $(2x^2y + 2y + 5)\,dx + (2x^3 + 2x)\,dy = 0$.

Solutions to the Problems on Elementary Solution Methods.

1. Solve: $(1+y^2)\,dx - x\,dy = 0$.

$$(1+y^2)\,dx - x\,dy = 0$$
$$(1+y^2)\,dx = x\,dy$$
$$\int \frac{1}{x}\,dx = \int \frac{1}{1+y^2}\,dy$$
$$\ln|x| + C = \arctan y$$
$$y = \tan(\ln|x| + C)\ \square$$

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2. Solve: $y' = \dfrac{2}{x}y + \dfrac{x^2}{\sqrt{1+x^2}}$.

Write the equation as
$$y' - \frac{2}{x}y = \frac{x^2}{\sqrt{1+x^2}}.$$
It's first-order linear. The integrating factor is
$$I = \exp\int -\frac{2}{x}\,dx = \frac{1}{x^2}.$$
Hence,
$$\frac{y}{x^2} = \int \frac{dx}{\sqrt{1+x^2}} = \ln|\sqrt{1+x^2} + x| + C.$$
Here is the work for the integral. Let $x = \tan\theta$, so $dx = (\sec\theta)^2\,d\theta$. Then
$$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{(\sec\theta)^2}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C = \ln|\sqrt{1+x^2} + x| + C.$$
The solution is
$$y = x^2(\ln|\sqrt{1+x^2} + x| + C).\ \square$$

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3. Solve: $(y\cos x - \cos x)\,dx + dy = 0$, $y(0) = 1$.

Rewrite the equation as
$$\frac{dy}{dx} + y\cos x = \cos x.$$
The equation is first-order linear. The integrating factor is
$$I = \exp\int \cos x\,dx = e^{\sin x}.$$
Hence,
$$ye^{\sin x} = \int \cos x\, e^{\sin x}\,dx = e^{\sin x} + C.$$

Therefore,

$$y = 1 + Ce^{-\sin x}.$$

Insert the initial condition:

$$1 = y(0) = 1 + C, \quad \text{so} \quad C = 0.$$

The solution is $y = 1$. $\square$

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4. Solve: $\dfrac{dy}{dx} = \dfrac{y}{2y \ln y + y - x}$, $y(2) = 1$.

The equation isn't first-order linear in y. Write the equation as

$$\frac{dx}{dy} = \frac{2y \ln y + y - x}{y} = 2 \ln y + 1 - \frac{1}{y} x, \quad \frac{dx}{dy} + \frac{1}{y} x = 2 \ln y + 1.$$

It's first-order linear in x. The integrating factor is

$$I = \exp \int \frac{1}{y}\, dy = y.$$

Therefore,

$$xy = \int (2y \ln y + y)\, dy = y^2 \ln y - \frac{1}{2} y^2 + \frac{1}{2} y^2 + C = y^2 \ln y + C.$$

Here's the work for the integral:

$$\begin{array}{cccc} & \dfrac{d}{dy} & & \int\, dy \\ + & \ln y & & y \\ & & \searrow & \\ - & \dfrac{1}{y} & \to & \dfrac{1}{2} y^2 \end{array}$$

$$\int y \ln y\, dy = \frac{1}{2} y^2 \ln y - \frac{1}{2} \int y\, dy = \frac{1}{2} y^2 \ln y - \frac{1}{4} y^2 + C.$$

The solution is

$$x = y \ln y + \frac{C}{y}.$$

Plug in the initial condition:

$$2 = 1 \cdot 0 + C, \quad C = 2.$$

The solution is

$$x = y \ln y + \frac{2}{y}. \ \square$$

---

5. Solve: $y' = \sin(x - y)$.

The equation isn't separable as it is. Let $z = x - y$, so $dz = dx - dy$. Eliminate y:

$$\frac{dy}{dx} = \sin(x - y)$$

$$dy = \sin(x - y)\, dx$$

$$dx - dz = \sin z\, dx$$

$$(1-\sin z)\,dx = dz$$
$$\int dx = \int \frac{1}{1-\sin z}\,dz$$
$$x + C = \tan z + \sec z$$
$$x + C = \tan(x-y) + \sec(x-y)$$

Here's the work for the z-integral:

$$\int \frac{1}{1-\sin z}\,dz = \int \frac{1}{1-\sin z}\,\frac{1+\sin z}{1+\sin z}\,dz = \int \frac{1+\sin z}{1-(\sin z)^2}\,dz =$$

$$\int \frac{1+\sin z}{(\cos z)^2}\,dz = \int \left(\frac{1}{(\cos z)^2} + \frac{\sin z}{(\cos z)^2}\right)\,dz = \int \left((\sec z)^2 + \sec z\tan z\right)\,dz = \tan z + \sec z + C. \;\square$$

---

6. Solve the equation $\dfrac{dy}{dx} = x^3(1-y)$, $y(0) = 3$. For what range of x values is the solution valid?

$$\frac{dy}{dx} = x^3(1-y)$$
$$\int \frac{1}{1-y}\,dy = \int x^3\,dx$$
$$-\ln|1-y| = \frac{1}{4}x^4 + C$$
$$1 - y = C_0 \exp\left(-\frac{1}{4}x^4\right)$$
$$y = 1 - C_0 \exp\left(-\frac{1}{4}x^4\right)$$

Insert the initial condition:

$$3 = y(0) = 1 - C_0, \quad \text{so} \quad C = -2.$$

Therefore,

$$y = 1 + 2\exp\left(-\frac{1}{4}x^4\right).$$

The solution is valid for all x. $\square$

---

7. Solve: $e^{-y}(1+y') = 1$.

$$1 + \frac{dy}{dx} = e^y$$
$$\frac{dy}{dx} = e^y - 1$$
$$\int \frac{1}{e^y - 1}\,dy = \int dx$$
$$\int \frac{e^{-y}}{1-e^{-y}}\,dy = x + C$$
$$\ln|1 - e^{-y}| = x + C \;\square$$

8. Solve: $(3x^2 - y^2)\,dx + \left(xy - \dfrac{x^3}{y}\right)\,dy = 0$.

Write the equation as

$$\frac{dy}{dx} = -\frac{3x^2 - y^2}{xy - \dfrac{x^3}{y}} = -\frac{3\dfrac{y}{x} - \left(\dfrac{y}{x}\right)^3}{\left(\dfrac{y}{x}\right)^2 - 1}.$$

The right side is a function of $\dfrac{y}{x}$. Let $v = \dfrac{y}{x}$, so $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. Substitute and separate variables:

$$v + x\frac{dv}{dx} = -\frac{3v - v^3}{v^2 - 1}$$

$$x\frac{dv}{dx} = -\frac{3v - v^3}{v^2 - 1} - v = \frac{-2v}{v^2 - 1}$$

$$\int \frac{v^2 - 1}{v}\,dv = -2\int \frac{dx}{x}$$

$$\int \left(v - \frac{1}{v}\right)\,dv = -2\int \frac{dx}{x}$$

$$\frac{1}{2}v^2 - \ln|v| = -2\ln|x| + C$$

$$\frac{1}{2}\left(\frac{y}{x}\right)^2 - \ln\left|\frac{y}{x}\right| = -2\ln|x| + C. \;\square$$

---

9. A 40-gallon tank is filled with pure water. Water containing 1 pound of dissolved salt per gallon enters at 4 gallons per minutes; the well-stirred mixture drains out at 8 gallons per minute. How much salt is in the tank when it is half-empty?

Let N be the number of pounds of salt dissolved in the water in the tank at time t.

$$\frac{dN}{dt} = \left(4\frac{\text{gal}}{\text{min}}\right)\left(1\frac{\text{lb}}{\text{gal}}\right) - \left(8\frac{\text{gal}}{\text{min}}\right)\left(\frac{N}{40 - 4t}\,\frac{\text{lb}}{\text{gal}}\right).$$

Simplify and rewrite the equation:

$$\frac{dN}{dt} = 4 - \frac{2}{10 - t}\,N, \quad \frac{dN}{dt} + \frac{2}{10 - t}\,N = 4.$$

It is first order linear in N. The integrating factor is

$$I = \exp\int \frac{2}{10 - t}\,dt = \frac{1}{(10 - t)^2}.$$

Therefore,

$$\frac{N}{(10 - t)^2} = \int \frac{4}{(10 - t)^2}\,dt = \frac{4}{10 - t} + C.$$

The solution is

$$N = 4(10 - t) + C(10 - t)^2.$$

There is no salt in the tank initially: When $t = 0$, $N = 0$:

$$0 = 40 + 100C, \quad C = -\frac{2}{5}.$$

Hence,

$$N = 4(10 - t) = -\frac{2}{5}(10 - t)^2.$$

The tank is losing 4 gallons per minute; it will be half-empty when it has lost 20 gallons, which occurs in 5 minutes. When $t = 5$, $N = 10$ pounds. $\square$

---

10. On what interval does the initial value problem

$$\frac{dy}{dx} + \frac{\sin x}{x} y = \frac{1}{x+5}, \quad y(-6) = 4$$

have a unique solution?

$\dfrac{\sin x}{x}$ and $\dfrac{1}{x+5}$ are continuous on the intervals $x < -5$, $-5 < x < 0$, and $x > 0$. The initial condition is $x = -6$, $y = 4$. Therefore, there is a unique solution on the interval $x < -5$. $\square$

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11. Solve: $(3x^2 - 2x - y)\,dx + (2y - x + 3y^2)\,dy = 0$.

$$M_y = -1 \quad \text{and} \quad N_x = -1.$$

The equation is exact. I need to find a function f such that $\dfrac{\partial f}{\partial x} = 3x^2 - 2x - y$ and $\dfrac{\partial f}{\partial y} = 2y - x + 3y^2$. Integrate M with respect to x:

$$f = \int (3x^2 - 2x - y)\,dx = x^3 - x^2 - xy + h(y)$$

Differentiate with respect to y:

$$2y - x + 3y^2 = \frac{\partial f}{\partial y} = -x + \frac{dh}{dy}$$

Therefore, $\dfrac{dh}{dy} = 2y + 3y^2$, so $h = y^2 + y^3$.
Hence, $f = x^3 - x^2 - xy + y^2 + y^3$. The solution is

$$x^3 - x^2 - xy + y^2 + y^3 = C. \square$$

---

12. (a) The equation

$$(2y - 5x - 7)\,dx + (2x - y + 3)\,dy = 0$$

is not homogeneous, because (e.g.) $-5x + 2y - 7$ is not homogeneous. However, it becomes homogeneous if you let

$$x = u + h, \quad y = v + k.$$

Perform this substitution and find values for h and k which make the constant terms in the coefficients of dx and dy vanish. Then solve the resulting equation.

Let $x = u + h$ and $y = v + k$, so $dx = du$ and $dy = dv$. Then

$$(-5u + 2v - 5h + 2k - 7)\,du + (2u - v + 2h - k - 3)\,dv = 0.$$

Next, I need to find h and k so that $-5h + 2k - 7 = 0$ and $2h - k - 3 = 0$. Solve the equations simultaneously to obtain $h = -1$ and $k = 1$.

With these values, the constant terms vanish, and the equation becomes

$$(-5u + 2v)\,du + (2u - v)\,dv = 0.$$

Rewrite the equation as

$$\frac{dv}{du} = -\frac{-5u + 2v}{2u - v} = \frac{-5 + 2\dfrac{v}{u}}{2 - \dfrac{v}{u}}.$$

The right side is a function of $\dfrac{v}{u}$. Let $w = \dfrac{v}{u}$, so $v = wu$ and $\dfrac{dv}{du} = w + u\dfrac{dw}{du}$. Substitute and separate variables:

$$w + u\frac{dw}{du} = -\frac{-5u + 2wu}{2u - wu}$$

$$u\frac{dw}{du} = -\frac{-5 + 2w}{2 - w} - w = \frac{5 - 4w + w^2}{2 - w}$$

$$\int \frac{2 - w}{w^2 - 4w + 5}\,dw = \int \frac{du}{u}$$

$$\frac{1}{2}\ln|w^2 - 4w + 5| = \ln|u| + C$$

$$\ln|w^2 + 2w + 2|^{1/2} = \ln|u| + C$$

$$|w^2 - 4w + 5|^{1/2} = C_0|u|$$

$$\left|\left(\frac{v}{u}\right)^2 - 4\left(\frac{v}{u}\right) + 5\right|^{1/2} = C_0|u|$$

$$|v^2 - 4uv + 5u^2|^{1/2} = C_0|u|^2$$

Now $u = x + 1$ and $v = y - 1$, so the solution is

$$|(y-1)^2 - 4(x+1)(y-1) + 5(x+1)^2|^{1/2} = C_0|x+1|^2. \ \square$$

(b) The equation

$$(8x + 4y + 1)\,dx + (4x + 2y + 1)\,dy = 0$$

appears to be similar to the equation in part (a), but the substitution used in part (a) won't work: You will find it impossible to find values for h and k which make the constant terms vanish.

The substitution fails because

$$8x + 4y = 4(2x + y) \quad \text{while} \quad 4x + 2y = 2(2x + y).$$

Take advantage of this fact by letting $z = 2x + y$. (Don't forget to substitute for the differentials!) Then solve the resulting equation.

Write the equation in the form

$$(4(2x+y)+1)\ dx + (2(2x+y)+1)\ dy = 0.$$

Set $z = 2x + y$, so $dz = 2\,dx + dy$, and $dy = dz - 2\,dx$. Then

$$(4z+1)\,dx + (2z+1)(dz - 2\,dx) = 0$$

$$-dx + (2z+1)\,dz = 0$$

$$\int dx = \int (2z+1)\,dz$$

$$x = z^2 + z + C$$

The solution is

$$x = (2x+y)^2 + (2x+y) + C. \ \square$$

(c) Can you solve $(8x+4y+1)\,dx+(4x+2y+1)\,dy = 0$ without using the method of homogeneous equations?

$$\frac{\partial M}{\partial y} = 4 \quad \text{and} \quad \frac{\partial N}{\partial x} = 4.$$

The equation is exact. I need to find a function f such that $\dfrac{\partial f}{\partial x} = 8x + 4y + 1$ and $\dfrac{\partial f}{\partial y} = 4x + 2y + 1$. Integrate M with respect to x:

$$f = \int (8x + 4y + 1)\,dx = 4x^2 + 4xy + x + g(y).$$

Differentiate with respect to y:

$$4x + 2y + 1 = \frac{\partial f}{\partial y} = 4x + \frac{dg}{dy}.$$

Then $\dfrac{dg}{dy} = 2y + 1$, so $g(y) = y^2 + y$. Hence, $f = 4x^2 + 4xy + x + y^2 + y$.
The solution is

$$4x^2 + 4xy + x + y^2 + y = C. \ \square$$

---

13. Solve: $(2x^2y + 2y + 5)\,dx + (2x^3 + 2x)\,dy = 0$.

$$M_y = 2x^2 + 2 \quad \text{and} \quad N_x = 6x^2 + 2$$

The equation is not exact. However,

$$\frac{M_y - N_x}{N} = \frac{-4x^2}{2x^3 + 2x} = \frac{-2x}{x^2 + 1}.$$

Since $\dfrac{M_y - N_x}{N}$ is a function of x alone, the integrating factor is

$$\mu = \exp \int \frac{-2x}{x^2+1}\,dx = \frac{1}{x^2+1}.$$

Multiply the original equation by $\dfrac{1}{x^2+1}$:

$$\left(2y + \frac{5}{x^2+1}\right)\,dx + 2x\,dy = 0$$

Now

$$M_y = 2 \quad \text{and} \quad N_x = 2.$$

Hence, the new equation is exact. I need to find a function f such that $\dfrac{\partial f}{\partial x} = 2y + \dfrac{5}{x^2+1}$ and $\dfrac{\partial f}{\partial y} = 2x$. Integrate M with respect to x:

$$f = \int \left(2y + \frac{5}{x^2+1}\right)\,dx = 2xy + 5\arctan x + h(y).$$

Differentiate with respect to y:

$$2x = \frac{\partial f}{\partial y} = 2x + \frac{dh}{dy}.$$

Therefore, $\dfrac{dh}{dy} = 0$ so $h = 0$.

Hence, $f = 2xy + 5\arctan x$. The solution is

$$2xy + 5\arctan x = C. \; \square$$