11-16-1993

The Divergence Theorem.

The **Divergence Theorem** is one of the "big" theorems of vector integral calculus. Let $\vec{F}$ be a 3-dimensional vector field, and D be a region in space. The Divergence Theorem relates the flux *outward* through the boundary ∂D of D to the triple integral of the divergence of $\vec{F}$ over D:

$$\int\int_{\partial D} \vec{F} \cdot \overrightarrow{dS} = \int\int\int_{D} \operatorname{div} \vec{F}\, dV.$$

("Outward" means that the surface integral is computed using the outward normal.)

The Divergence Theorem is related to Stokes' theorem and the Fundamental Theorem of Calculus. In fact, they are all special cases of an abstract "Stokes' theorem" which holds in arbitrary dimensions. The table below shows how these results are related.

Dimension	**Result**
1	$F(b) - F(a) = \int_a^b \frac{d}{dx} F(x)\, dx$
2	$\int\int_S \operatorname{curl} \vec{F} \cdot \overrightarrow{dS} = \int_{\partial S} \vec{F} \cdot \overrightarrow{ds}$
3	$\int\int_{\partial D} \vec{F} \cdot \overrightarrow{dS} = \int\int\int_D \operatorname{div} \vec{F}\, dV$

- In dimension 1, you have the **Fundamental Theorem of Calculus**.
- In dimension 2, youhave **Stokes' theorem**. (I could have included **Green's theorem** on this line, since it is just Stokes' theorem in the plane.)
- In dimension 3, you have the **Divergence Theorem**.

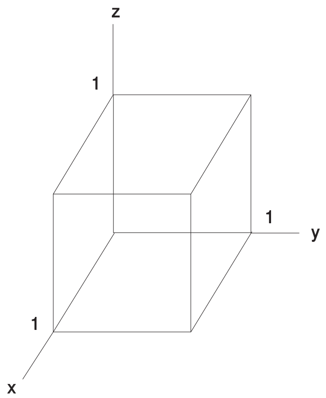
"Dimension" refers to the dimension of the object you're integrating over (on the right side). In the Fundamental Theorem, you integrate over a segment $[a, b]$. In Stokes' theorem, you integrate over a surface S. In the Divergence Theorem, you integrate over a region D.

In each case, the integrand on the right involves a "derivative" of the integrand on the left. Finally, each of the integrals on the left is computed over the **boundary** of the region for the integral on the right. Symbolically, the results all look like

$$\int_{\partial R} \sigma = \int_R d\sigma.$$

(You can make this statement a bit more precise via the theory of **differential forms**.)

Example. Find the flux of $\vec{F} = \langle 6xy, 2y^3 + y, 2xz \rangle$ out of the unit cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.



To compute the flux directly, you'd need to compute the flux through each of the 6 faces, and then add up the results. Rather than do 6 computations, it's better to use the Divergence Theorem.

First, the divergence is

$$\operatorname{div} \vec{F} = 6y + 6y^2 + 1 + 2z.$$

The inequalities for the region are

$$0 \le x \le 1, \quad 0 \le y \le 1, \quad 0 \le z \le 1.$$

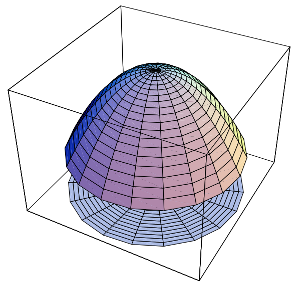
By the Divergence Theorem,

$$\int\int_{\partial D} \vec{F} \cdot \overrightarrow{dS} = \int_0^1 \int_0^1 \int_0^1 (6y+6y^2+1+2z)\, dx\, dy\, dz = \int_0^1 \int_0^1 (6y+6y^2+1+2z)\, dy\, dz = \int_0^1 (6+2z)\, dz = 7. \square$$

Remark. The Divergence Theorem is often useful in situations like this one, where the boundary of the region is made up of several pieces. $\square$

Example. Compute the flux of $\vec{F} = \langle -2xz, 2yz - y^2, z^2 + 3yz \rangle$ upward through $z = 1 - x^2 - y^2$, $z \ge 0$.

Notice that this surface *doesn't* bound a region in space, so I can't apply the Divergence Theorem "as is". However, it will bound a region if I include the disk $x^2 + y^2 \le 1$, $z = 0$.



Apply the Divergence Theorem to the region enclosed by the paraboloid together with the disk. In cylindrical coordinates, the region is

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 1, \quad 0 \le z \le 1 - r^2.$$

Now div $\vec{F} = y + 2z$, so by the Divergence Theorem,

$$\int\int_{\partial D} \vec{F} \cdot \overrightarrow{dS} = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} (2z + r\sin\theta)\, dz\, r\, dr\, d\theta =$$

$$\int_0^{2\pi} \int_0^1 \left(r(1-r^2)^2 + r^2(1-r^2)\sin\theta\right)\, dr\, d\theta = \int_0^{2\pi} \left(\frac{2}{15}\sin\theta + \frac{1}{6}\right)\, d\theta = \frac{\pi}{3}.$$

This represents the *total flux* through the boundary of the region, which is

$$\text{flux through paraboloid} + \text{flux through disk}.$$

Since I want the flux through the paraboloid, I must subtract the flux *downward* through the disk from the result I just obtained. (*Downward*, because this is *out of* the region.)

The disk is

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 1, \quad z = 0.$$

Hence, the normal vector is $\langle 0, 0, -1\rangle$. Then

$$\vec{F} \cdot \vec{N} = z^2 + 3yz = 0.$$

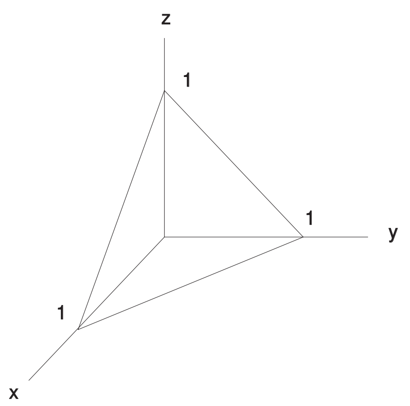
(Remember that $z = 0$ for the disk!) The flux through the bottom is 0, so the flux through the paraboloid is $\dfrac{\pi}{3} - 0 = \dfrac{\pi}{3}$. □

Remark. When you apply the Divergence Theorem, be sure you're applying it to the *boundary of a region*, i.e. a *closed surface*. If you want to apply it to a non-closed surface, you must "close the surface up" with extra pieces of surface, as I did in the last example. Then you will need to compensate by subtracting off the flux through the pieces you added.

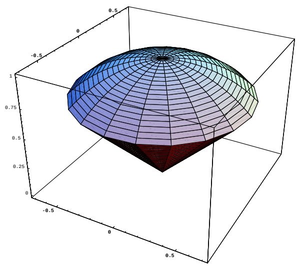
Because you will need to compute the flux through the pieces you added, this method is not always that useful. You may be better off computing the flux through the original surface directly, without using the Divergence Theorem. □

Problems on the Divergence Theorem.

1. Find the flux of $\vec{F} = \langle x^2 - xyz, 3y + \frac{1}{2}y^2 z, 2yz - 3z\rangle$ outward through the boundary of the region R in the first octant cut off by the plane $x + y + z = 1$.

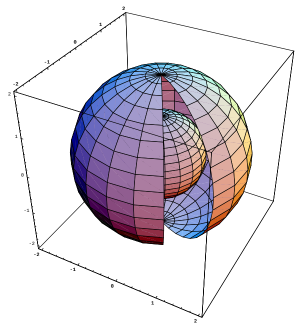


2. Let R be the region defined by $\sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2}$. Find the flux of $\vec{F} = \langle 2-6xz^2-2xy^2, yz^2+y^3, 2z^3+x^2z\rangle$ outward through the boundary of R.



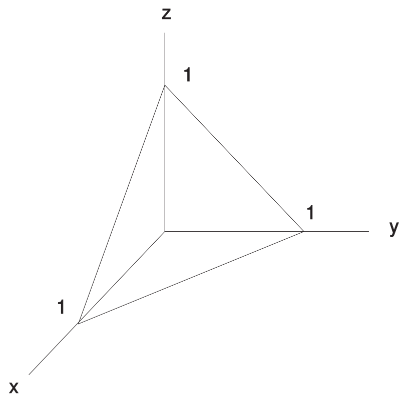
3. Find the flux of $\vec{F} = \langle 3-y^2+x, 2xy-y^2, 2yz-2xz\rangle$ out of the region between the spheres

$$x^2+y^2+z^2=1 \quad \text{and} \quad x^2+y^2+z^2=4.$$



Solutions to the Problems on the Divergence Theorem.

1. Find the flux of $\vec{F} = \langle x^2 - xyz, 3y + \frac{1}{2}y^2 z, 2yz - 3z \rangle$ outward through the boundary of the region R in the first octant cut off by the plane $x + y + z = 1$.



I'll apply the Divergence Theorem, since computing the flux directly would mean computing the flux through each of the 4 faces of the tetrahedron.

First, $\operatorname{div} \vec{F} = 2x + 2y$.

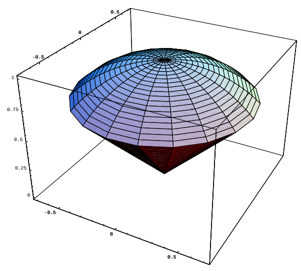
The region R is determined by the inequalities

$$0 \le x \le 1, \quad 0 \le y \le 1 - x, \quad 0 \le z \le 1 - x - y.$$

So

$$\iint_{\partial D} \vec{F} \cdot \overrightarrow{dS} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (2x + 2y)\, dz\, dy\, dx = \frac{1}{6} \approx 0.16667. \ \square$$

2. Let R be the region defined by $\sqrt{x^2 + y^2} \le z \le \sqrt{1 - x^2 - y^2}$. Find the flux of $\vec{F} = \langle 2 - 6xz^2 - 2xy^2, yz^2 + y^3, 2z^3 + x^2 z \rangle$ outward through the boundary of R.



The region may be described in spherical coordinates by

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{4}, \quad 0 \le \rho \le 1.$$

The divergence is

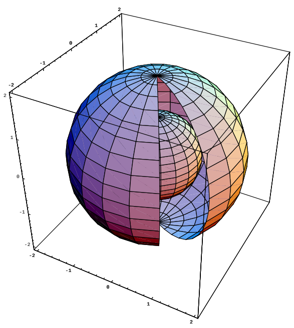
$$\operatorname{div} \vec{F} = x^2 + y^2 + z^2 = \rho^2.$$

By the Divergence Theorem,

$$\iint_{\partial D} \vec{F} \cdot \overrightarrow{dS} = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^1 \rho^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{2\pi}{5} \int_0^{\frac{\pi}{4}} \sin\phi \, d\phi = \frac{2\pi}{5}\left(1 - \frac{\sqrt{2}}{2}\right) \approx 0.36806. \square$$

3. Find the flux of $\vec{F} = \langle 3 - y^2 + x, 2xy - y^2, 2yz - 2xz \rangle$ out of the region between the spheres

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x^2 + y^2 + z^2 = 4.$$



The divergence is

$$\operatorname{div} \vec{F} = 1 + 2x - 2y + 2z - 2x = 1 - 2y + 2z.$$

The region D between the two spheres may be described in spherical coordinates by

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi, \quad 1 \le \rho \le 2.$$

The spherical conversion equations give

$$y = \rho \sin\phi \sin\theta, \quad z = \rho\cos\phi.$$

Hence, by the Divergence Theorem,

$$\iint_{\partial D} \vec{F} \cdot \overrightarrow{dS} = \int_0^{2\pi} \int_0^{\pi} \int_1^2 (1 - 2\rho\sin\phi\sin\theta - \rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta =$$

$$\int_0^{2\pi} \int_0^{\pi} \left(\frac{7}{3}\sin\phi - \frac{15}{2}(\sin\phi)^2 \sin\theta - \frac{15}{4}\sin\phi\cos\phi\right) d\phi \, d\theta =$$

$$\int_0^{2\pi} \left(\frac{14}{3} - \frac{15}{4}\pi\sin\theta\right) d\theta = \frac{28\pi}{3} \approx 29.32153. \square$$